Can unpredictable risk exposure be priced?

Barahona, Ricardo Driessen, Joost Frehen, Rik*

January, 2018

Abstract

In this paper we study the pricing of betas that are hard to predict. We argue that in a world where investors cannot perfectly predict betas, the degree of beta predictability should matter for the extent to which these betas are priced. We show this is in a model with ambiguity averse investors that are uncertain about an asset's risk exposure to an exogenous risk. By taking the perspective of an investor, we provide empirical evidence to support this model, where downside, size and book-to-market betas which are hard to predict, are also not priced when we use betas that would be available to investors when these make their decisions.

Keywords: Risk factors, Beta, Ambiguity aversion, Risk hedging
JEL classification: G11, G12

1 Introduction

While many factors have been proposed in the asset pricing literature to explain the cross section of stock returns, little attention has been paid to how well investors can observe the risk exposure to these factors. While standard asset pricing models assume that betas and risk exposures are known, in this paper we look at asset pricing from the perspective of an investor that is not able to perfectly observe the exposure of an asset to given risk factor. When investors use the stock market to hedge risks, it is important for them to know an asset's risk exposure in order to effectively use it as a hedging instrument. However, if investors cannot observe risk exposures, then how can they care enough that these risk exposures are priced? In the extreme case where risk exposures are unpredictable, then there should be no demand associated to these risk exposures and they should not be priced. We first develop a simple model with exogenous risks and find that

*All authors are from the Finance Department at Tilburg University.

investors exposed to an exogenous risk have lower absolute demands for a hedging asset as the asset's exposure to the exogenous risk becomes more uncertain. In turn, this drives expected returns of the hedging asset towards zero. Empirically we show that downside, size and book-to-market betas are difficult to forecast, and that betas available to investors at the date of the investment decision are not priced.

We start by solving a model with two investor types, where one type is exposed to an exogenous risk. Both investors have access to a zero net supply asset that can potentially hedge this risk, however its correlation with the exogenous risk is not directly observed. Without any uncertainty in hedging capability, the investor exposed to exogenous risks will be willing to pay the other investor in order reduce the risk exposure, while the other investor is willing to provide insurance and reap a return premium. However, if the investor exposed to exogenous risk has ambiguity aversion and cannot observe how well the hedging asset is able to provide an effective hedge, then they will want lower amounts of this insurance, driving the demand for this asset towards zero. As a result, the lower demand drives the expected returns of the hedging asset towards zero in absolute terms.

In practice, the hedging portfolio can be thought of as a self financing portfolio available for agents to invest in, that is long securities with high exposure to the exogenous risk and short securities with low exposure. The exogenous risk is a reduced form way of modeling additional risks orthogonal to market risk and the hedging portfolio is a factor mimicking portfolio that agents can invest in. Uncertainty around the hedging effectiveness is then uncertainty around the true beta of this portfolio with respect to the exogenous risk [1]. If investors are unable to form portfolios that are exposed to exogenous risks ex-post, the result is that exposure to exogenous risks should result in lower expected returns in absolute terms.

Empirically, we provide evidence that this effect may be important when pricing stocks. We specifically look at *downside risk* (Ang et al. (2006a)) and the two additional risk factors from the Fama and French (1992) 3 factor model, SMB (small-minus-big) and HML (high-minus-low) risk[2]. We empirically estimate a model to forecast stock betas and find that exposures to these risk factors are highly unpredictable, raising the question of how relevant beta loadings on these risks should be in the cross section of returns. When using risk exposures available to investors at the portfolio formation date, we find either lower risk prices in

[1]To be more precise, we model uncertainty around the correlation and not the beta, however both are related.

[2]SMB and HML risk is a security's beta exposure to long-short portfolios that go long small firms while shorting large firms and long high book to market while shorting low book to market firms respectively. These portfolios are assumed proxy for additional risks in addition to aggregate market risk. It should be noted however, that Daniel and Titman (1997) has spurred a large literature that contests whether these portfolios do indeed proxy for risk.

the best case scenario, or that risks are not priced at all. Furthermore, we use a portfolio choice framework to empirically estimate investor demands for portfolios and find that investors with potential hedging demand for these 3 risks do not wish to invest long-short portfolios constructed based on risk exposure estimates available at the portfolio formation date.

Related to this paper is the Harvey et al. (2016) critique on the enormous amount of factors proposed in the literature to explain the cross section of returns. In response to this, Harvey et al. (2016) to suggest raising the bar when looking at *t-statistics* of additional anomalies due to the possible publishing bias resulting from the large amount of research studying this question. Given the many attempts by researchers to find additional factors that can explain the cross section of returns (both published and unpublished attempts), Harvey et al. (2016) show that we can no longer use the standard *t-tests* in order to determine if a factor is indeed statistically significant. To address this issue, they devise a multiple testing framework, where as the research community tests more factors, the threshold for a *t-test* to be statistically significant increases as well.

While Harvey et al. (2016) might help us reduce the number of factors that investors should consider in practice, we provide an additional dimension to solve this issue. One problem that has been less explored is that many of these models work under the assumption that investors observe the true risk loadings of assets, when in fact they are estimated from the data, often with considerable estimation error. Furthermore, in a setting where asset risk exposures are time varying, knowing past risk exposures may provide insufficient information regarding future risk exposures. Indeed, we find it hard to construct long-short portfolios with reliable beta exposure to the three risks we measure. This underlines our main question, if risk parameters are time-varying and unobservable to investors, uncertainty around the risk estimates should be an important determinant of the risk price.

Throughout the many risk factors proposed in the asset pricing literature, we focus on three. We start with *downside beta* as defined by Ang et al. (2006a). Downside risk is related to crash risk has recently become popular in financial literature since the rare disasters model was re-introduced by Barro (2006), and extended by Gabaix (2012) and Wachter (2013) to explain several puzzles in financial economics such as the equity premium and excess variance puzzles. When looking at crash risk in the cross section of returns, crash risk arises if some assets are more sensitive to crashes than others, even when controlling for market risk. Ang et al. (2006a) empirically investigate this with their measure for crash risk, downside beta. Simply put, downside beta is the market beta of a security estimated over the days where the market under-performs. When controlling for market betas, stocks with high downside betas are expected to perform poorly during market crashes. Ang et al. (2006a) empirically show that stocks with high downside betas have higher expected returns. Crash risk has also been empirically investigated using options data. While Santa-Clara and Yan (2010) find that aggregate market jump risk may be responsible for as much as 40% of te equity risk premium (as opposed to diffusion risk or volatility which explains the remainder of the premium), Cremers et al. (2015) find that there is also a cross sectional premium for stocks with a high exposure to aggregate market jump risk.

However, one of the main assumptions for these models is that investors know the true parameters that define a stock's exposure to that source of risk. This assumption is especially relevant for the case of crash or downside risk, where it is empirically challenging to determine an asset's exposure to this kind of risk ex-ante. In fact, we find that when sorting stocks into portfolios using ex-ante measures of crash risk, we are unable to find any significant dispersion in the crash risk of these portfolios, making it especially puzzling that there is empirical evidence finding a pricing relationship between crash risk exposure and asset returns. Cremers et al. (2015) briefly address this issue when they find that despite the cross sectional premiums obtained when sorting stocks according to their ex-post risk loadings on aggregate jump risk, they no longer find a premium when sorting using ex-ante measures such as past exposures or forecasts using additional individual firm data.

While crash risk may be an important source of risk, since the introduction of the capital asset pricing model (CAPM) by Sharpe (1964), Lintner (1965), Black (1972) and others, there has been a wide literature extending this model to settings where asset prices are also affected by factors other than crash risk. Our paper is related to many of these extensions since they all require that investors know an asset's risk loading on a certain source of risk ex-ante, and just like crash risk, it is not at all clear that risk loadings for additional factors are easy for investors to know ex-ante. For this reason we also investigate the betas from the popular CAPM extension, the Fama-French 3 factor model introduced by Fama and French (1992). In addition to downside beta, we also look at the SMB and HML betas from this model.

This paper is organized as follows, in section 2 we develop our theoretical model to illustrate how parameter uncertainty around asset correlations affects asset prices, section 3 we investigate beta predictability, in section 4 we compare asset pricing results when using contemporaneous and predictive beta measures, in section 5 we analyze investors' optimal demands and in section 6 we conclude.

2 Ambiguity aversion and correlation uncertainty

There are only few papers that look at ambiguity aversion in relation to portfolio choice, most papers focus on the uncertainty of expected returns on assets. The uncertainty of expected returns of assets has been shown to provide theoretical explanations such as low participation in equity markets (Easley and O'Hara (2009)) and home bias (Maccheroni et al. (2013)). Maenhout (2004) and Gollier (2011) have also shown how ambiguity aversion and model uncertainty can account for a high equity premium while decreasing demands for equities. However these do not directly look at how ambiguity aversion can affect the cross section of asset prices.

One paper comes close, Garlappi et al. (2007) use ambiguity aversion around mean estimates of stocks in order to investigate optimal ways of constructing portfolios. This paper modifies the standard mean variance problem to include ambiguity aversion around stock means in order to improve portfolio decisions. However, Garlappi et al. (2007) do not focus on how this might affect asset prices. Furthermore, they only look at ambiguity aversion for the first moment of stock distributions.

However, many of the risk factors that explain the cross section of stock returns arise from hedging demands. Hedging demands come from investors' exposure to a variety of risks that are exogenous to the market. One example we focus on is crash risk. Investors may be exposed to crash risk due to a variety of reasons, such as lower worse investment opportunities or higher risk of income loss (via unemployment for example). Assets positively correlated with these negative outcomes are then more risky, since they provide investors with bad outcomes when they face bad states. Assets that are less correlated however can provide some form of insurance against these exogenous risks. The correlations and covariances of returns are then parameters of large importance regarding return distributions. However, these are unobservable parameters that investors have to estimate and that may vary over time. If this is the case, investors should care about how well they can estimate and predict the covariance structure of assets over the period in which they invest in. To model this, we introduce investors that have ambiguity aversion with respect to the covariance of assets and exogenous risks.

In order to better understand how uncertainty affects asset prices, we develop a simple model with exogenous untradable risks. This will be our reduced form way of introducing hedging demands for the representative investor. Another important ingredient in this model is the introduction of ambiguity aversion. We model ambiguity aversion as in Garlappi et al. (2007), where agents who are uncertain about a parameter in their utility function, choose a value for this parameter within a certain confidence interval that minimizes their utility. When choosing their portfolio, given a certain confidence interval, agents choose the worst case scenario within that interval.

This model is our first step in order to understand how the predictability of risk may affect the premium associated with that risk. In the asset pricing literature, many of the asset pricing models that predict cross sectional differences in assets' expected returns motivate this as a result of asset hedging demands [3]. This demand arises from a group of risk averse investors that are exposed to some non aggregate market risk, using the available assets at their disposable to hedge these risks to a certain degree.

One possible example is that some investors might have strong preferences to hedge crash risk. These could be large institutional investors such as banks, pension funds or insurance companies that may be subject to regulatory pressure that might make them avoid certain risky assets. In order to avoid regulatory pressures, institutional investors may by willing hedge themselves against crash risk going short assets that are highly exposed to crashes. For the case of stocks, in practice this would mean these investors could short a zero cost portfolio that loads on crash risk, while investors who care less about crash risk could be willing to buy these assets, receiving some compensation for bearing this extra risk.

However, in an environment where exposure to crash risk is time-varying and unobservable, we argue that an agent's ability to forecast this exposure is also important to determine prices. In this specific example, it is hard to tell which stocks are exposed to crash risk ex-ante. Cremers et al. (2015) for example show that stock loadings on the JUMP factor, a factor that proxies for the risk of jumps in market returns, are hard to predict. In this situation, agents might not be willing to give up as much return for assets that they are uncertain about regarding their risk hedging potential. Our model will explore how uncertainty around the correlation between an asset and an exogenous risk, i.e. uncertainty about the hedging potential, affects asset prices compared to a model where agents observe the true correlations.

Given a myriad of models that predict cross sectional differences in asset expected returns generated not only by their market beta but also investor's hedging demands, we develop a model that analyses how uncertainty around the correlation of traded assets and a given exogenous risk affects asset prices.

[3] Any model that uses factors motivated by the Merton (1973) ICAPM model would fit this description.

2.1 Market model with a zero net supply portfolio

2.1.1 Assets

In our model, economic agents can trade 2 risky assets, the market portfolio, M, and a hedging portfolio, H. They can also have exposure to a non-traded asset or exogenous risk, Q. In total, there are 3 risks M, H and Q, where M and H are traded. The excess returns of the 3 risky assets have the following joint distributions:

$$r_{t+1} = \begin{bmatrix} r_{M,t+1} \\ r_{H,t+1} \\ r_{Q,t+1} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_M \\ \mu_H \\ \mu_Q \end{bmatrix}, \begin{bmatrix} \sigma_M^2 & 0 & 0 \\ 0 & \sigma_H^2 & \sigma_{QH} \\ 0 & \sigma_{QH} & \sigma_Q^2 \end{bmatrix} \right), \tag{1}$$

where we can alternatively write σ_{QH} as $\sigma_{QH} = \sigma_Q \sigma_H \rho$, where ρ is the correlation coefficient between Q and H.

The market asset must be in positive net supply. Since this is the only traded asset in positive supply, the total market capitalization must be equal to the total traded wealth in the economy, W_T, which we normalize to 1. The hedging portfolio is in zero net supply, which means the wealth all agents invest in this asset must be zero. Asset H is orthogonal to the market, while having some correlation ρ to Q. We can think of H as a self financing market neutral portfolio of assets that has some positive exposure to the exogenous risk, while the market as a whole is independent of this exogenous risk. In this case, investing in or shorting H can be seen as a deviation from the market portfolio. Given these 3 assets, the expected returns of M and H will be endogenous to the model, while the expected return of Q and the variance-covariance matrix of the risky assets is given.

2.1.2 Agents

In this model, we have 2 types of agents, $j \in \{A, B\}$, with risk aversion γ^j where agent A is not exposed to exogenous risks, $w_Q^A = 0$, while agent B has a positive exposure to the exogenous risk, $w_Q^B > 0$. Furthermore, agents of type A and B detain all the wealth in the economy and each own a relative share of wealth W_j. They must invest a portion of their wealth in each of the 2 traded assets: the market asset M, and the hedge asset H. Since we write everything in terms of returns in excess of the risk free rate, these weights don't need to add to 1. Excess portfolio returns are then given by:

$$r_{p,t+1}^j = w_M^j r_{M,t+1} + w_H^j r_{H,t+1} + w_Q^j r_{Q,t+1}, \tag{2}$$

with expected return

$$E(r^j_{p,t+1}) = w^j_M \mu_M + w^j_H \mu_H + w^j_Q \mu_Q, \tag{3}$$

and variance

$$Var(r^j_{p,t+1}) = (w^j_M)^2 \sigma^2_M + (w^j_H)^2 \sigma^2_H + (w^j_Q)^2 \sigma^2_Q + 2 w^j_H w^j_Q \sigma_H \sigma_Q \rho. \tag{4}$$

Finally we introduce the notion of ambiguity aversion around the ρ parameter, the correlation coefficient between the hedge asset, H, and the exogenous risk, Q. We model this as in Garlappi et al. (2007), where in our case agent B doesn't know the true value of ρ, but has a point estimate, $\hat{\rho}$, and a confidence interval such that with some confidence level it is known that $\rho \in [\hat{\rho} - \eta; \hat{\rho} + \eta]$. Given this interval, an ambiguity averse agent will then choose an estimate for ρ within this confidence interval such that it minimizes the agent's utility, i.e. the agent makes a decision on the worst case scenario of ρ. This is the same notion of ambiguity aversion preferences of Gilboa and Schmeidler (1989), where agents maximize their utility over the worst case scenario for the parameter sensitive ambiguity aversion. In our framework, a higher η can either represent more ambiguity aversion or a higher standard error of the estimate. However this does not alter our conclusions, if we consider a constant ambiguity aversion, higher η means higher correlation uncertainty. Furthermore, ρ will only enter agent B's utility since it is the only agent exposed to exogenous risks, therefore ambiguity aversion is only relevant for agent B.

In our model, each agent wants to maximize their next period expected utility which we model using a quadratic utility function. Summarizing this information and inserting it into our agents' quadratic preferences, the objective functions for agent A is then given by

$$\max_{\{w^A_M, w^A_H\}} U^A = w^A_M \mu_M + w^A_H \mu_H - \frac{\gamma^A}{2}((w^A_M)^2 \sigma^2_M + (w^A_H)^2 \sigma^2_H) \tag{5}$$

and agent B by

$$\begin{aligned} \max_{\{w^B_M, w^B_H\}} \min_{\{\rho\}} U^B = {} & w^B_M \mu_M + w^B_H \mu_H + w_Q \mu_Q \\ & - \frac{\gamma^B}{2}((w^B_M)^2 \sigma^2_M + (w^B_H)^2 \sigma^2_H + (w_Q)^2 \sigma^2_Q + 2 w^B_H w_Q \sigma_H \sigma_Q \rho). \quad (6) \\ s.t. \quad & \rho \leq \hat{\rho} + \eta \\ & \rho \geq \hat{\rho} - \eta \end{aligned}$$

For simplicity, since $w_Q^A = 0$, we redefine $w_Q \equiv w_Q^B$.

2.1.3 Equilibrium conditions

Since we consider both agents to be price takers, market equilibrium is achieved once both agents optimally allocate to the available assets given their current prices, μ_M and μ_H. We start by solving the model for the case without uncertainty, where $\hat{\rho} = \rho$ and $\eta = 0$. The partial equilibrium demands of each investor are described in the proposition that follows.

Proposition 1. *If there is no uncertainty such that $\hat{\rho} = \rho$ and $\eta = 0$, then the optimal partial equilibrium demands for agents A and B are given by*

$$\begin{cases} w_M^{A*} = \frac{\mu_M}{\gamma^A \sigma_M^2} \\ w_H^{A*} = \frac{\mu_H}{\gamma^A \sigma_H^2} \end{cases} \tag{7}$$

and for B,

$$\begin{cases} w_M^{B*} = \frac{\mu_M}{\gamma^B \sigma_M^2} \\ w_H^{B*} = \frac{\mu_H}{\gamma^B \sigma_H^2} - \frac{w_Q \sigma_H \sigma_Q \rho}{\sigma_H^2} \end{cases} \tag{8}$$

Proof: See appendix.

In full equilibrium, the markets for both tradable assets have to clear. This means the total wealth allocated to the market portfolio has to add up to a positive amount which we normalize to 1. On the other hand, the total wealth invested in the hedge asset should be zero since it is in zero net supply. These two additional conditions can be written as:

$$W_A w_M^A + W_B w_M^B = W_T = 1, \tag{9}$$

and

$$W_A w_H^A + W_B w_H^B = 0. \tag{10}$$

Solving for these for these conditions, yields the propositions that follow.

Proposition 2. *If there is no uncertainty such that $\hat{\rho} = \rho$ and $\eta = 0$, then the market premium and hedge asset premiums are given by*

$$\begin{aligned} \mu_M &= \bar{\gamma}\sigma_M^2 \\ \mu_H &= \bar{\gamma} W_B w_Q \sigma_H \sigma_Q \rho \end{aligned} \tag{11}$$

where $\bar{\gamma}$ is the wealth weighted harmonic average of the risk aversion coefficients of both agents.

Proof: See appendix.

Since the market is uncorrelated to either the background risk or the hedge asset, the expected return of the market depends on the level of market risk and the average risk aversion level of the agents.

The premium of the hedge asset however, is driven by the average risk aversion, the share of wealth of the type B agent, her exposure to the background risk and the covariance between the hedge asset and the background risk. From here, we can see that part of the premium is driven by the hedging demand of agent B, where the agent's available wealth and exposure to the background risk increase the premium in absolute terms. The other important component is the correlation of the hedge asset and the untradable risk. In absolute terms, higher correlations result in higher premiums, while the sign of the premium will depend on whether the correlation is negative or positive. It is important to see that a correlation of zero would make the premium disappear.

The next step involves adding ambiguity aversion. For this we return to agent B's objective function in equation (6). While in the previous case agent B is assumed to know ρ, in this case the agent observes an estimate, $\hat{\rho}$ and a confidence interval such that $\rho \in [\hat{\rho} - \eta; \hat{\rho} + \eta]$ with an certain degree of confidence. The size of the interval, η, will then depend on both the degree of confidence required by the agent as well as the standard error of the estimate.

In this setup, ambiguity aversion shows up with the agent choosing a ρ^* within that interval that minimizes the agent's utility, as opposed to using the observed estimate $\hat{\rho}$. In order to obtain the equilibrium under ambiguity aversion, we now now solve the equilibrium conditions as before, with the difference that the objective function for agent B is now different. While the demands of agent A and the demand for the market asset of agent B remain the same, under ambiguity aversion, the demand of agent B for the hedge asset will now by given by the proposition that follows.

Proposition 3. *Under ambiguity aversion, the partial equilibrium demand of agent B for the hedge asset and the optimal ρ will be given by the following expressions.*

$$\rho^* = \begin{cases} \hat{\rho} + \eta & ,\hat{\rho} < -\eta \\ 0 & ,\hat{\rho} - \eta \leq 0 \leq \hat{\rho} + \eta \\ \hat{\rho} - \eta & ,\hat{\rho} > \eta \end{cases} \tag{12}$$

$$w_H^{B*} = \begin{cases} \dfrac{\mu_H}{\gamma^B \sigma_H^2} - \dfrac{w_Q \sigma_H \sigma_Q (\hat{\rho} + \eta)}{\sigma_H^2} & ,\hat{\rho} < -\eta \\ 0 & ,\hat{\rho} - \eta \leq 0 \leq \hat{\rho} + \eta \\ \dfrac{\mu_H}{\gamma^B \sigma_H^2} - \dfrac{w_Q \sigma_H \sigma_Q (\hat{\rho} - \eta)}{\sigma_H^2} & ,\hat{\rho} > \eta \end{cases} \tag{13}$$

The partial equilibriums demands of agent A remain unchanged as do those of agent B for the market.

Proof: See appendix.

From the proposition above, we can see that the demand for the hedge asset from the agent exposed to the exogenous risk shrinks towards zero if uncertainty around the correlation, η, is high. This comes as a result from the agent's ambiguity aversion. The ambiguity aversion drives agent B to choose the ρ^* within a the given confidence interval that minimizes the agent's utility and as we can see from the proposition, this adjustment implies a smaller ρ^* in absolute terms. If the observed correlation is above zero, agent B will revise that estimate and shrink it towards zero and consider the correlation at the lower boundary of the confidence interval. The opposite is true when the estimate is below zero, agent B will revise the correlation estimate at the upper bound of the interval. Finally, when the estimate is close enough to zero (zero is within the confidence interval), then the agent will consider a correlation of zero.

Intuitively, this means that whenever the confidence interval for $\hat{\rho}$ includes zero, an investor will make decisions based on the worst case scenario, which happens when ρ is zero and the investor is unable to hedge against exogenous risk. If this is not the case, then the correlation considered for agent B's optimal portfolio demands is always lower in absolute terms than what is observed. This means the investor makes decisions considering the case where the hedge asset provides less hedging potential due to its lower correlation in absolute terms with the exogenous risk. These demands lead to the equilibrium returns described in the proposition below.

Proposition 4. *By inserting the optimal demands of both agents into the market clearing conditions from equation (10), we obtain the following equilibrium returns for the hedge asset.*

$$\mu_H = \begin{cases} \bar{\gamma} W_B w_Q \sigma_H \sigma_Q (\hat{\rho} + \eta) & , \hat{\rho} < -\eta \\ 0 & , \hat{\rho} - \eta \leq 0 \leq \hat{\rho} + \eta \\ \bar{\gamma} W_B w_Q \sigma_H \sigma_Q (\hat{\rho} - \eta) & , \hat{\rho} > \eta \end{cases} \tag{14}$$

Proof: See appendix.

From this proposition, we that the ambiguity aversion of agent B will drive the risk premium of the hedge asset towards zero in absolute terms. Notice that the hedge asset premium is linear in ρ^*, this means that agent B's ambiguity about the hedge asset's hedging potential is what drives the absolute returns of the hedging asset to be smaller. Investors not exposed exogenous risk will in turn want less of the hedge asset due to the asset's lower return.

We can clearly see this in Figure 1. There we calibrate the model and plot the optimal ρ of agent B and the expected returns of the hedge asset in a model with and without ambiguity aversion against the observed $\hat{\rho}$. In Panel B we can see how the agent shrinks the optimal correlation towards zero. This effect is larger as we move away from zero and is solely dependent on the size of the confidence interval. When looking at the Panels C where we plot the equilibrium returns and D, where we plot the absolute equilibrium returns we can see a similar effect in equilibrium returns. In absolute terms returns decrease compared to the model without ambiguity aversion the further the observed $\hat{\rho}$ is from zero.

While this model provides a simplified way for us to think about uncertainty in risk, we may also infer what the results might imply more generally for the cross section of stocks. In practice these results suggest that in a conditional multi factor pricing model where beta loading of individual stocks are time-varying, it is important for investors to accurately predict future beta loadings. The implication here is that when betas are hard to predict and have large forecast errors, then the price of the respective factors should be lower in absolute terms. If it is hard to form portfolios exposed to beta risks ex-ante, then an investor's optimal demand for such a portfolio should be small.

It is easy to see why we need ambiguity aversion for uncertainty about the correlation to matter. If we were to set η to zero, this would provide us with a result similar to as if there was no uncertainty. Investors, would simply base their decisions on $\hat{\rho}$, without correcting for the fact that it is an estimate and not the true correlation, leading to the same results as in propositions 1 and 2. Furthermore, it is important to note that in this model, we only study the impacts of uncertainty covariances between assets and non-market risks, that is, betas of factors other than the market. In fact, Gollier (2013) suggests that market beta uncertainty of an asset increases risk premiums. For this reason we do not intend to focus on market betas, instead we focus on betas that are extensions of the CAPM. Intuitively speaking, this arises from the fact that the market is in positive net supply while self self financing portfolios that provide hedges are in zero net supply, so uncertainty works differently in this case.

3 Predicting Betas

In order to show the importance of properly estimating risk factors for assets we construct a dataset using CRSP daily common stock returns and COMPUSTAT yearly firm accounting data starting in 1963 through 2016, similar to those of Ang et al. (2006a) and Cremers et al. (2015). We follow most of the literature in selecting NYSE, AMEX and NASDAQ stocks with CRSP share codes 10 and 11. In order to estimate betas, we follow Cremers et al. (2015) using daily data over the course of a year in order to estimate a stock's yearly beta every month. From our sample, we exclude stocks that don't have corresponding accounting data on COMPUSTAT. We also exclude stocks that at the beginning of each year have prices lower than 5 USD, to avoid stocks that are illiquid or may have extreme returns due to their low prices. Furthermore, when estimating betas, we consider that stocks that have more than 5 missing daily returns over the course of the one year window to be missing observations. We also follow Fama and French (1993) by removing financial firms from the sample as firm fundamental variables such as the book to market ratio and leverage have different interpretations for these firms. Finally, we split our sample at the beginning of every year into the 20% largest firms and 80% smallest firms by market cap. While we find that predicting betas is quite difficult, this problem is smaller for larger firms where there is likely less noise in returns. While our results will focus on the sample of large stocks, in a separate appendix we also provide the results for the sample of the 80% smallest stocks, where we find similar results.

3.1 Betas

In this paper, we look at four different betas. We look at the market beta, downside beta from Ang et al. (2006a), and SMB and HML betas from Fama and French (1993). Since we are using daily returns and there are several smaller illiquid stocks in our sample, we use the Dimson (1979) correction for non-synchronous trading. For the CAPM beta, we regress stock returns on the market portfolio of the same day and the previous day as in equation (15). The final beta estimate is the sum of β_1 and β_2.

$$r_{i,t} = \alpha_i + \beta_1 r_{m,t} + \beta_2 r_{m,t-1} + e_{i,t} \tag{15}$$

We use the relative downside beta (Beta DR henceforth) as in Ang et al. (2006a) because standard downside beta is highly correlated to the standard market beta. What we call standard downside beta, is calculated as the market beta where the estimation sample is restricted to only include the days in which the market return was below its average during the estimation window. In our setup, this means we estimate a stock's market beta using the only the days in which the stock market had a return below its average over the past year.

$$\beta_i^{Downside} = \frac{Cov(r_{i,t}, r_{m,t} | r_{m,t} < \mu_m)}{Var(r_{m,t} | r_{m,t} <] \mu_m)} \tag{16}$$

However, since by construction this measure includes half of the sample used to estimate a normal market beta, these two measures are highly correlated. Since we only want to capture the downside risk component, we follow Ang et al. (2006a) and we subtract the market beta from the standard downside beta in order to obtain a stock's relative downside beta. Ang et al. (2006a) show that stocks that load on Beta DR risk don't necessarily load on market beta risk, however the same is not true for standard downside beta.

In order to compute Beta DR with the Dimson correction, we first have to calculate the standard downside beta. To do that, we run the regression from equation (15), however from the yearly return window, we drop all observations where the market performed better than the average market return for that one year window. Therefore, we only perform the regression on the observations where the market had a lower return than the average return of the estimation window. The downside beta is then the sum of β_1 and β_2, and in order to obtain beta DR, we subtract the market beta from this.

For SMB and HML betas we perform a similar operation to the CAPM beta regression, in addition to the market, we regress on the factor return on the same day as well as the previous day and add both coefficients β_3 and β_4 from equation (17) in order to get the final factor estimate.

$$r_{i,t} = \alpha_i + \beta_1 r_{m,t} + \beta_2 r_{m,t-1} + \beta_3 r_{k,t} + \beta_4 r_{k,t-1} + e_{i,t} \tag{17}$$

3.2 Beta Prediction

While in many cross sectional studies, researchers have looked at the relationship between risk and returns contemporaneously[4], we argue that risk exposures should only be priced if they are predictable to a certain degree. Indeed, Cremers et al. (2015) find the puzzling result that despite the fact that stocks exposed to jump risk have higher returns, they are not able to identify these stocks ex-ante. The main focus of this paper is to then study the implications of the ability to forecast risk and the relationship with returns and investor demands.

Instead of using contemporaneous risk factors, we want to draw a relationship between expected risk and expected return. One way is to simply look at past risk measures and expected returns, however, we can do better than simply use past risk measures. Instead we can try and predict future risk exposures using firm fundamental data. In order to forecast next period betas with information that is available ex-ante, we largely follow Boyer et al. (2010).

In order to forecast betas, we estimate the following cross-sectional regression every month:

$$\beta_{i,t+12} = \delta_0 + \delta_1\beta_{i,t} + \delta_2 X_{i,t} + e_{i,t+12}, \tag{18}$$

where $\beta_{i,t}$ represents the most recently estimated beta, which uses the past 12 months of data, $X_{i,t}$ is a vector of predictive variables. We use these variables to forecast betas over the next 12 months which means there is no overlap between the estimation of the betas on the right and left hand sides. To reduce the influence of outliers, we winsorize the bottom and top 2.5th percentiles of observations every month. Furthermore, to help us in terms of interpretation, we also standardize the right hand side variables every month.

Our goal is to use the latest available betas along with firm fundamental data in order to forecast betas over the next one year. Besides using betas estimated in the past year, we also use a set of fundamental variables constructed with data from CRSP and COMPUSTAT. Using CRSP returns we also estimate coskewness and idiosyncratic volatility over the past year. Like the betas, we estimate coskewness and idiosyncratic volatility every month over the past 1 year window. Using the last available 1 year of daily data, we estimate coskewness as:

$$CSK_i = \frac{E[(r_i - \mu_i).(r_m - \mu_m)^2]}{\sqrt{Var(r_i)}.Var(r_m)}, \tag{19}$$

[4] Ang et al. (2006a) and Cremers et al. (2015) for example.

following Harvey and Siddique (2000), where μ_i and μ_m are the stock i's and the market means over the past year using daily data. Idiosyncratic volatility is also calculated using daily returns stock excess returns over the past year. To estimate the idiosyncratic volatility, we calculate the volatility of daily excess returns in excess of the market defined as:

$$\epsilon_{i,t} = r_{i,t} - \beta_i^M r_{m,t}, \tag{20}$$

where β_i^M is the market beta of stock i estimated over the past year, as explained in the previous section.

We also use firm fundamental data in order to predict future betas. From CRSP, we extract firm market capitalizations at the end of each month. We then use COMPUSTAT yearly accounting data to construct additional variables. Like in Fama and French (1993), we are conservative and assume that data from fiscal year t is only available starting July from calendar year t and is used every month until June from calendar year $t+1$. We use the Book-to-market (BTM) ratio, calculated as the ratio of last available book equity to the end of month market capitalization. Return on equity (ROE) is calculated as the ratio of net income to book equity. We follow Campbell et al. (2008) and calculate leverage as the ratio of book debt to the sum of book debt and market capitalization. Finally, we use Fama and French (1996) industry definitions to group SIC codes into 12 industry dummies.

Table 1 shows us the average coefficients of these monthly cross sectional regressions, with Fama and MacBeth (1973) standard errors for the average coefficients. Due to the overlapping windows of each cross section, we also correct standard errors using 12 Newey-West lags.

[insert Table 1 about here]

The main takeaways from these regressions are that it seems quite difficult to predict betas out of sample. Our prediction model seems to explain market betas quite well when compared to Beta DR, Beta SMB and Beta HML given the higher average adjusted R^2's. This seems to come mostly from the fact that market betas are more persistent than the other betas as we can see from the lagged beta coefficients. Past Beta DR although significant, has a coefficient close to zero meaning that it is a poor predictor of future Beta DR. The SMB beta is still quite persistent when compared to beta HML and Beta DR, however less persistent than market Beta, however this is not significant across all specifications. Beta HML on the other hand seems to oscillate given a large negative coefficient on the lagged beta. This suggests that stocks with high HML Beta today are likely to have the lowest betas in the cross section over the following year. Furthermore, size and BTM seem to be important predictors of Beta SMB and Beta HML respectively which is not surprising given the way the factors are constructed.

To evaluate how well each model does in forecasting future betas, we also calculate the average root mean square errors that the forecasts from this model produce in the following way:

$$RMSE = \frac{\sum_{t=1}^{T} \sqrt{\sum_{i=1}^{N} \left(\beta_{i,t} - E[\beta_{i,t}]\right)^2 / N_t}}{T}, \tag{21}$$

where N_t is the number of stocks in a given month, β_t is the realized beta over the past 12 months, and $E[\beta_t]$ is the out of sample forecast beta generated from the most average recent δ coefficients from equation (18) and most recent available data prior the beta estimation window. Since predictions are out of sample, an investor can use these betas to make investment decisions.

Along with the lagged beta coefficients, the root mean square errors of our forecasts highlight how unstable these betas can be over time, and that they are relatively hard to predict. While we start by looking at predictability using only lagged betas, we can see that the models under the full specification using characteristics and industry dummies do a slightly better job in predicting future betas. We will therefore use betas estimated from the fully specified model later on in this paper and refer to them as *forecast betas*.

It is important to note that the betas we look at have different cross sectional distributions so the RMSE's are not directly comparable across different betas. However, in the following section, we explain how we analyze long short portfolios constructed using beta forecasts, giving us a better idea of how useful these can be in a way that is somewhat comparable across betas.

4 The Pricing of Contemporaneous and Predictive Betas

Our analysis is focused on whether risks are priced in a multifactor model of the following form:

$$E_t(r^i_{t+1}) = \beta^M_{i,t}\lambda_{m,t} + \sum_{k=1}^{K} \beta^k_{i,t}\lambda_{k,t}, \tag{22}$$

where $\beta_{i,t}^M$ and $\lambda_{m,t}$ represent a stock's market beta and the price of market risk respectively and $\beta_{i,t}^k$ and $\lambda_{k,t}$ represent additional stock betas and the respective prices of risk. The betas in this equation, are the asset betas conditional on the information available at time t, which cannot be perfectly observed by an econometrician. To deal with this issue, the finance literature has generally followed three distinct approaches. The first is to use the realizations of beta at time $t+1$, which corresponds to the same window used to calculate returns. The second uses pre-ranking betas, beta estimates that are available at time t, that is, estimated before time $t+1$. The underlying assumption is that current betas are good proxies for the conditional covariances of assets. A third and final approach is to forecast betas using information available at time t, such as firm accounting data and firm specific risk measures. Researchers resort to this alternative if they believe that betas at time t do not accurately reflect the true conditional betas. The first method, which relies on realizations of betas, we call contemporaneous betas since they are estimated in the same window as returns, while the second and third we call predictive since they rely exclusively on information available before the returns of the following period are realized.

We are particularly interested in predictive betas. From an investor's point of view, these are the betas they are able to use in order to make informed investment decisions, while contemporaneous betas are only available ex-post. We specifically focus on the relationship between $\beta_{i,t}^k$ and expected returns. While our model suggests that expected betas matter for the pricing relationship, so does the degree of certainty we have around this estimate. In our model, we document that when correlations between an asset and a risk factor are not observed but instead estimated with a certain degree of uncertainty, then the price of risk should be smaller in absolute terms. In practical terms, this suggests that if investors are unable to predict future betas associated with a certain risk factor k, then this factor should not be priced in the cross section of stocks.

4.1 Portfolio Sorts

To investigate these relationships, our first step to identify whether betas are priced is to create quintile portfolios based upon realized betas. For each of the betas (Beta DR, Beta SMB and Beta HML), we construct portfolios based on realized, predicted and pre-ranking beta sorts. For the realized beta sorts, every month, we rank stocks according to their estimated betas over the past 12 months and then use these betas to group stocks into value weighted quintile portfolios. Using these portfolio formations, we look at their performance over the same 12 month window used to estimate the betas.

In Table 2, we can see the results from this analysis in the realized beta panels.

While for Beta DR and market beta there is a contemporaneous relationship between beta risk and return as in Ang et al. (2006a), the same is not true for SMB and HML betas. When forming a long-short portfolio that goes long the high beta quintile portfolio and shorts the low beta portfolio, we find that portfolios sorted on Beta DR show positively significant returns even when adjusted for market risk, given the high and positively significant CAPM alpha. Furthermore, returns increase monotonically every quintile. However, we don't find this monotonic pattern for portfolios formed on realized SMB and HML betas, with the addition that the long short portfolios constructed on these two measures have returns that are not statistically significant from zero over the sample.

[insert Table 2 about here]

However, we are interested in the relationship between expected betas and returns. In order to do this, we use two alternative measures of expected betas. Instead of comparing betas and returns over the same window, we construct portfolios in two different ways. We first use pre-ranking betas. In this case, we use the direct beta estimated made over the past 12 months to form portfolios and measure the returns of these portfolios over the next 12pre-ranking betas, we use predicted betas generated from our prediction model in the previous section in order to form value weighted portfolios. The predictions are made using information available over the past 12 months, and we then form quintile portfolios based on these predictions and measure their returns over the following 12 months.

While the portfolios formed on realized betas are created with information that is not available ex-ante, the information to construct portfolios formed using pre-ranking and predicted betas is. This means that from an investor's point of view, realized beta portfolios are not investable, as opposed to the predicted and pre-ranking betas portfolios which are fully investable.

Unlike when using realized betas, we no longer find a monotonic relationship between portfolio betas and returns, and the long-short portfolio displays returns that are not significantly different than zero. We further highlight that the dispersion of realized Beta DR in the quintile portfolios is much smaller when compared the realized beta sorts. While the realized Beta DR of the long-short portfolio is positive and significant, it is much smaller than when using realized betas. Altogether, these results show that it is difficult for an investor to use pre-ranking betas in order to gain exposure to Beta DR risk and reap the potential premium identified in the realized beta sorts. Similar results are found when using predicted betas instead, however, when using predicted betas, we find a positively significant yearly alpha of 4.37%. While this might suggest that our prediction model is performing well, we prefer to interpret this with caution for two reasons. First, the dispersion in Beta DR among the 5 portfolios is similar to the pre-ranking beta, meaning that our prediction model still struggles to identify which stocks are exposed to Beta DR risk ex-post. This raises a second concern, given that the realized Beta DR of the portfolios sorted on predicted betas are quite similar, the return differential might be due to other factors. In fact, our prediction model uses several variables that have been associated in the literature to explaining the cross section of stock returns[5]. Since the predicted betas are linear combinations of these, our concern is that our results come from the fact that we are indirectly sorting on variables such as idiosyncratic volatility.

Regarding SMB and HML betas, our results are different than for Beta DR. For both of these betas, there is a large dispersion in portfolio realized betas, whether sorting on pre-ranking or predicted betas, with the betas of the long short portfolio being positive and significant. However results on the returns premiums when comparing the high and low beta portfolios are mixed. For SMB beta, the long-short portfolios fail to capture positively significant alphas. While we note that when using predicted SMB beta there is a positively significant premium when looking at raw returns, once we control for market risk, this premium disappears. As for HML beta, the premium of the long-short portfolios seem to be positive but insignificant for pre-ranking beta. Regarding predicted HML beta, there seems to be a positively significant CAPM alpha, however the same caution in interpretation applies as for Beta DR.

To further investigate the predictability of betas, we also analyze the time-series distribution of the realized betas of the long-short portfolios constructed using pre-ranking and predicted betas. The objective here is to see whether we can use past data to reliably gain exposure to these beta risks. For each beta, we have a set of overlapping long-short portfolios, formed every month with a 1 year investor horizon over our whole sample. We can look at the distribution of these betas over the sample to construct confidence intervals to investigate what exposure to each beta can be expected for a given year.

From Table 3, we can see that depending on the required confidence interval, an investor is not guaranteed positive exposure to beta risk. For the case of Beta DR, whether using pre-ranking or predicted betas, investors may face negative downside betas when investing in a long-short portfolio, even when requiring a 90% confidence interval. This means that for this confidence level, this portfolio could have the opposite exposure of what an investor would aim to have over a given year. If an investor hoping to hedge against downside risk would short this portfolio, this means that there is a considerable chance that they would in fact get even more exposure to downside risk. Furthermore, we note that the

[5]Fama and French (1992) find size and book to market explain the cross section of returns, while Harvey and Siddique (2000) and Ang et al. (2006b) find that coskewness and idiosyncratic volatility explain the cross section of stock returns respectively.

prediction model does little to shorten these confidence intervals.

For SMB and HML betas, the prediction model somewhat helps achieving the desired positive exposure when considering the 99% confidence interval meaning that the prediction model has some power to reduce some of the more extreme observations. However, at confidence intervals of 95% or lower, predicted and pre-ranking betas both succeed in providing investors investing in a long-short portfolio with positive exposures to these risks. Nevertheless, we noted that while investors may be able to achieve positive risk exposures with a certain degree of confidence, the confidence intervals are also very wide. While this suggests that investors that wish to be exposed to SMB or HML risks may be able to do so, controlling the amount of exposure however may be much harder.

[insert Table 3 about here]

Drawing from our theoretical framework, these large beta confidence intervals should play a role in the pricing of these risks. These empirical results suggest that while it may be possible for investors to use the stock market to take care of their hedging needs, there is a high degree of uncertainty to this. The amount of uncertainty should be tied to the size of these premiums. If stocks are unable to reliably satisfy the hedging needs of investors, then risks that arise from these hedging demands should carry a lower price. These results are in line with the low R^2 and high RMSE of the prediction models from Table 1, as well as the small dispersion in betas of the pre-ranking and predicted beta portfolios. Furthermore, we can see in the descriptive statistics of Table 4, that Beta DR, SMB and HML all have very low average autocorrelations across stocks, which provides further evidence on the unpredictability of these betas.

4.2 Pricing Regressions

In this section, we further investigate the pricing relationship between betas and returns by estimating cross sectional regressions. We use Fama and MacBeth (1973) regressions to estimate the risk prices attached to realized betas and compare them to those when using predicted betas. Table 4 shows the summary statistics of the variables used in these regressions.

[insert Table 4 about here]

These allow us to better interpret the economic significance of some of our results since in many cases our regressions our run using standardized variables, so that we can compare results across different betas. Furthermore, we can also get a better idea of how hard in can be to predict future betas. Looking at panel B from the descriptive statistics, we can see that the the average time-series autocorrelation for stock betas seems to be very low.

In order to quantify the prices for the beta risks, the general specification of the regression that we estimate in Panel A of Table 5 is the following:

$$r_{i,t} = \lambda_{m,t}\beta_{i,t}^{M} + \sum_{k=1}^{K} \lambda_{k,t}\beta_{i,t}^{k} + \gamma_t X_{i,t-1} + \varepsilon_{i,t}, \tag{23}$$

where $\beta_{i,t}^{M}$ and $\beta_{i,t}^{k}$ are the betas estimated over the same one year window as $r_{i,t}$, and $X_{i,t-1}$ is a vector of control variables. Our coefficients of interest are the $\lambda_{k,t}$, the price of the non market beta risks. In Panels B and C, we run a similar regression, however we replace realized betas with predicted betas, that we estimated using information at $t-1$. Since realized and predicted betas have different cross sectional distributions, when running cross sectional regressions, we standardize the right hand side variables for every period in order easily make comparisons between the different betas. Furthermore, due to the way predicted betas are constructed, we are not able in include controls for the asset pricing regressions of predicted betas in Panel C. Since these betas are constructed using linear combinations of these control variables, this results in collinearity problems when estimating these regressions.

[insert Table 5 about here]

Here we wish to compare the coefficients in Panel A and to those of Panels B and C. For Beta DR, we provide empirical evidence that predicted betas carry a lower price of risk than the realized betas. When used alone (column 1 of panels A, B and C), there is roughly a large reduction in the expected return of a stock for an increase in one standard deviation of realized and predicted beta. When using pre-ranking betas, Beta DR is not priced. Results for pre-ranking Beta DR remain the same when adding controls, with the risk price being insignificantly different than zero. However, Beta DR it is positively priced when using predicted betas, albeit with a 33% reduction in the risk price when compared to the contemporaneous betas in Panel A. However, this difference is much smaller and almost negligible when adding other betas to the regression. For Panel C, where Beta DR is regressed alone against returns is in line with the results from Table 2, once we add additional betas to the regression, predicted beta and realized betas seem to carry similar risk premiums. Still, the realized betas seem to carry higher returns when looking at the specifications that include all betas across Panels A and C. In Panel C column (7) for example, downside beta is no longer different than zero at a 95% confidence level, whereas in Panel A for the same specification (column 10), downside beta seems is priced and significantly different than zero.

In general these results paint a picture where betas that are available to investors at the investment date do not explain differences in expected returns across stocks. If we combine this with the low predictability of Beta DR analyzed in the previous sections, this is in line with our theoretical predictions, where unpredictable betas should not be priced.

Results for Beta SMB and HML allow us to reach similar conclusions. While the realized and pre-ranking betas seem to have little relationship with returns, this changes once we use predicted betas. We should note however that these betas also seem to be highly unstable, as shown in Tables 1 and 3. Especially for the case of HML betas, results from Table 1 suggest that these betas oscillate a lot, since stocks that have high HML betas seem to display much lower betas in the following period and vice versa. However, we should still take results from predicted betas with a grain of salt. As explained earlier, predicted betas are estimated from variables that have been suggested in the literature to explain the cross section of returns. Since we cannot use controls in Panel C due to multicollinearity issues, we cannot rule out that the it is the combination of characteristics and not the actual predicted betas that are explaining expected returns.

While the results for predicted betas in Panel C are not entirely clear, we find that when using pre-ranking betas in Panel B, Beta DR, SMB and HML do not seem to be priced. When looking at the predicted betas from Panel C, for Beta DR, predicted betas seem to either have a lower premium or no premium at all, whereas for SMB and HML there seems to be a premium as opposed to what we observe in Panels A and B.

5 Beta portfolios from the investor's perspective

Another prediction from our model comes from Proposition 3, where high beta uncertainty should lead to lower partial equilibrium demands for the hedge asset. Following Driessen and Maenhout (2007), we use a framework where we empirically estimate the optimal weights to allocate to a set of assets for a CRRA investor with varying degrees of risk aversion. We frame the problem so that the investor has the choice of investing in the market portfolio and an additional long short portfolio, which we call the hedge portfolio.

In our general setting, investors that want to maximize the utility expected utility of their terminal wealth,

$$\max_{\alpha_i} E\left[U\left(W_T\right)\right], \tag{24}$$

where terminal wealth is given by

$$W_T = W_0\left[R_f + \alpha_E\left(R_E - R_f\right) + \alpha_H R_H\right]. \tag{25}$$

Here $i \in E, H$, where E and H refer to returns and weights on the equity market portfolio and the hedge portfolio respectively. The hedge portfolios we investigate are the long-short portfolios we form in the previous section based on downside, SMB and HML beta quintile sorts. In this analysis, we consider both the portfolios formed on realized/contemporaneous betas as well the portfolios formed predictively using pre-ranking and predicted betas.

To estimate the optimal weights, we construct portfolios with a yearly investment horizon at the start of every month as in the previous section. We then use GMM to estimate the optimal unconditional weights. The moment conditions can be extracted by taking the first order conditions of equation (24), which yield:

$$\begin{cases} E\left[U'\left(W_T\right)\left(R_E - R_f\right)\right] = 0 \\ E\left[U'\left(W_T\right)\left(R_H\right)\right] = 0 \end{cases} \tag{26}$$

While this specification allows for several kinds of preferences, we use CRRA preferences. This preference class is particularly appropriate for downside risk since it is able to model concern with higher order moments regarding risk preferences. Linking back to our theoretical model, CRRA preferences penalize skewness in returns, that is, CRRA investors dislike crashes. While in our model we write this in the reduced form of a mean variance investor with an aversion to abstract exogenous risk, here that exogenous risk comes from the CRRA preferences. The dislike for negatively skewed returns (like crashes) of this type of investor, should generate some hedging demand towards downside risk where we would expect negative demands for the long-short portfolio of downside beta. However, if the premium for crash risk is high, then the opposite may be true. Using CRRA utility yields the following moment conditions:

$$\begin{cases} E\left[W_T^{-\gamma}\left(R_E - R_f\right)\right] = 0 \\ E\left[W_T^{-\gamma}\left(R_H\right)\right] = 0 \end{cases} . \tag{27}$$

[insert Table 6 about here]

We use GMM to estimate the weights that satisfy these moment conditions. Since we have overlapping portfolios, we correct standard errors for autocorrelation with Newey-West standard errors with 12 lags. In Table 6, we estimate the optimal weights for CRRA investors with a level of risk aversion that ranges from 1 to 10, in 5 different settings which correspond to Panels A to E respectively:

A Only the market portfolio (model without R_H and α_H);

B The market portfolio and a hedge portfolio constructed using realized betas (the same portfolios from Table 3). This portfolio is not investable in practice.

C The market portfolio and a hedge portfolio constructed using the most recent beta estimates at the formation date. This portfolio is investable in practice.

D The market portfolio and a hedge portfolio constructed using the most recent beta forecasts. This portfolio is investable in practice.

E The market portfolio, a hedge portfolio formed using predicted downside beta and SMB and HML portfolios as controls.

From Table 6, we find that the availability of the hedge portfolios, does not make a CRRA investor shift away from the market portfolio, as optimal weights on the market remain similar in panels B to E. This is consistent with the fact that the hedge portfolios have very low betas and are almost orthogonal to the market portfolio.

For hedge portfolios constructed with contemporaneous betas, we find that a CRRA investor would only want invest in the relative downside beta risk, choosing to have a positive exposure to this risk. Since CRRA preferences are sensitive to higher order moments and downside risk, it is surprising that such an investor would positively load on this risk, and even more the fact that this investor would allocate more to to downside risk than to equity. This is possibly explained by the fact that the premium for downside beta seems large when analyzed using realized betas.

Regarding panels C, D and E, where the hedge portfolios are constructed using available data and therefore fully investable, we find investors with CRRA preferences do not choose to allocate to these portfolios at levels statistically significant than zero, except for the case of investors with very low risk aversion. Even for downside beta, although when using a hedge portfolio constructed with predicted betas (Panel D) we find positively significant weights, this result disappears once we include investable characteristic based portfolios into the investors available choices. This suggests that predicted downside beta partly picking up value and size anomalies, since our prediction is a linear combination of factors that include these two variables.

Furthermore, we note that although in some cases weights are not statistically significant, they are generally positive for all the hedge portfolios we test, at all risk aversion levels. This is also surprising as all these portfolios are in zero net supply, suggesting that we require investors with different kinds of preferences to explain these sorts of cross sectional pricing differences across stocks.

6 Conclusion

In this paper we look at asset pricing from the perspective of an investor. While in most of the asset pricing literature investors are assumed to observe asset risk exposures, we find that this is far from a trivial task. While there have been steps in the literature to improve beta estimates[6], in a setting where risk is time-varying, it is important to be able to accurately forecast betas or risk exposures for the investment horizon over which the agent wants to invest their money.

We show this theoretically in a setting where investors are not able to observe the exact joint distributions of the available assets. Investors willing to give up return in order to reduce exposure to an exogenous risk, dislike not knowing the joint distribution of the exogenous risk and the assets that could hedge it. Intuitively it makes sense as investors are unsure of whether an asset provides hedging potential or not, will be less willing to give up returns for this unsure prospect.

We begin to empirically investigate the predictions from this model by quantifying just how unpredictable downside, SMB and HML betas are. Using our prediction best models, we indeed find that this a non trivial task and that we obtain very high forecast errors. In practice this is translated into how difficult it is to construct portfolios out of sample that go long in high beta stocks and short in low beta stocks, that manage to achieve a consistent positive exposure to the betas used for sorting.

We find that when using pre-ranking betas, there is no evidence for premiums associated with downside, HML and SMB betas. However, when analyzing beta forecasts from our prediction model, there is some pricing evidence for all three betas. When looking at downside beta, while there is some evidence that forecast downside betas are priced, the risk prices are somewhat smaller when compared to the prices of realized bet downside beta, which is unobservable to investors.For

[6]See Vasicek (1973), Karolyi (1992) and Cosemans et al. (2015).

SMB and HML betas, there is some evidence that predicted betas are priced, however with the caveat that we are not able to properly control for characteristics. Furthermore we find that when empirically modeling an investor's portfolio decision problem, there is little demand for portfolios based on risk exposures estimated predictively.

Finally, we also find evidence that CRRA preferences are not enough to explain the cross section of asset prices as they consistently overweight zero-net supply portfolios constructed based on some of the documented anomalies in the asset pricing literature. While it may be that economic agents with CRRA like preferences exist, we require additional agents to take the other side of these trades. This is not the main focus of our paper, so we leave this for future research.

References

Ang, A., Chen, J., and Xing, Y. (2006a). Downside risk. *Review of Financial Studies*, 19(4):1191–1239.

Ang, A., Hodrick, R. J., Xing, Y., and Zhang, X. (2006b). The cross-section of volatility and expected returns. *The Journal of Finance*, 61(1):259–299.

Barro, R. J. (2006). Rare disasters and asset markets in the twentieth century. *The Quarterly Journal of Economics*, 121(3):823–866.

Black, F. (1972). Capital market equilibrium with restricted borrowing. *The Journal of Business*, 45(3):444–455.

Boyer, B., Mitton, T., and Vorkink, K. (2010). Expected idiosyncratic skewness. *Review of Financial Studies*, 23(1):169–202.

Campbell, J. Y., Hilscher, J., and Szilagyi, J. (2008). In search of distress risk. *The Journal of Finance*, 63(6):2899–2939.

Cosemans, M., Frehen, R., Schotman, P. C., and Bauer, R. (2015). Estimating security betas using prior information based on firm fundamentals. *The Review of Financial Studies*, 29(4):1072–1112.

Cremers, M., Halling, M., and Weinbaum, D. (2015). Aggregate jump and volatility risk in the cross-section of stock returns. *The Journal of Finance*, 70(2):577–614.

Daniel, K. and Titman, S. (1997). Evidence on the characteristics of cross sectional variation in stock returns. *the Journal of Finance*, 52(1):1–33.

Dimson, E. (1979). Risk measurement when shares are subject to infrequent trading. *Journal of Financial Economics*, 7(2):197–226.

Driessen, J. and Maenhout, P. (2007). An empirical portfolio perspective on option pricing anomalies. *Review of Finance*, 11(4):561–603.

Easley, D. and O'Hara, M. (2009). Ambiguity and nonparticipation: The role of regulation. *The Review of Financial Studies*, 22(5):1817–1843.

Fama, E. F. and French, K. R. (1992). The cross-section of expected stock returns. *the Journal of Finance*, 47(2):427–465.

Fama, E. F. and French, K. R. (1993). Common risk factors in the returns on stocks and bonds. *Journal of financial economics*, 33(1):3–56.

Fama, E. F. and French, K. R. (1996). Multifactor explanations of asset pricing anomalies. *The journal of finance*, 51(1):55–84.

Fama, E. F. and MacBeth, J. D. (1973). Risk, return, and equilibrium: Empirical tests. *Journal of political economy*, 81(3):607–636.

Gabaix, X. (2012). Variable rare disasters: An exactly solved framework for ten puzzles in macro-finance. *Quarterly Journal of Economics*, 127(2):645–700.

Garlappi, L., Uppal, R., and Wang, T. (2007). Portfolio selection with parameter and model uncertainty: A multi-prior approach. *Review of Financial Studies*, 20(1):41–81.

Gilboa, I. and Schmeidler, D. (1989). Maxmin expected utility with non-unique prior. *Journal of mathematical economics*, 18(2):141–153.

Gollier, C. (2011). Portfolio choices and asset prices: The comparative statics of ambiguity aversion. *The Review of Economic Studies*, 78(4):1329–1344.

Gollier, C. (2013). Asset pricing with uncertain betas: A long-term perspective.

Harvey, C. R., Liu, Y., and Zhu, H. (2016). and the cross-section of expected returns. *The Review of Financial Studies*, 29(1):5–68.

Harvey, C. R. and Siddique, A. (2000). Conditional skewness in asset pricing tests. *The Journal of Finance*, 55(3):1263–1295.

Karolyi, G. A. (1992). Predicting risk: Some new generalizations. *Management Science*, 38(1):57–74.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *The review of economics and statistics*, pages 13–37.

Maccheroni, F., Marinacci, M., and Ruffino, D. (2013). Alpha as ambiguity: Robust mean-variance portfolio analysis. *Econometrica*, 81(3):1075–1113.

Maenhout, P. J. (2004). Robust portfolio rules and asset pricing. *Review of financial studies*, 17(4):951–983.

Merton, R. C. (1973). An intertemporal capital asset pricing model. *Econometrica: Journal of the Econometric Society*, pages 867–887.

Santa-Clara, P. and Yan, S. (2010). Crashes, volatility, and the equity premium: Lessons from s&p 500 options. *The Review of Economics and Statistics*, 92(2):435–451.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The journal of finance*, 19(3):425–442.

Sion, M. (1958). On general minimax theorems. *Pacific Journal of mathematics*, 8(1):171–176.

Vasicek, O. A. (1973). A note on using cross-sectional information in bayesian estimation of security betas. *The Journal of Finance*, 28(5):1233–1239.

Wachter, J. A. (2013). Can time-varying risk of rare disasters explain aggregate stock market volatility? *The Journal of Finance*, 68(3):987–1035.

Appendix

A Proof Proposition 1 and 2

Proof. In order to reach equilibrium, two conditions need to be met: i) both agents must choose their optimal portfolios and ii) the markets for assets M and H need to clear. For a given levels of risk aversion, expected payoffs, variances and covariances of the available assets, agents will trade and decide in equilibrium the portion of their wealth, w_M^j and w_H^j, that they invest in each asset and at what prices, μ_M and μ_H, they will trade them.

By taking derivatives of each agent's utility with respect to each asset weight, we obtain the optimal portfolios of both agent's described in proposition 1.

Given the fixed supply of the market and hedge assets, the 2 agents will trade them until supply is exhausted (supply = demand). Since the market asset is the only asset in positive net supply, all wealth must be invested in asset M.

$$W_A w_M^A + W_B w_M^B = W_T = 1 \tag{28}$$

As for the hedge asset it is in zero net supply, so in aggregate, the total amount of wealth in this economy invested in H is zero.

$$W_A w_H^A + W_B w_H^B = 0 \tag{29}$$

Plugging in the optimal weights of each agent, we can solve the market clearing conditions for the expected returns of M and H and obtain

$$\mu_M = \bar{\gamma}\sigma_M^2 \tag{30}$$

$$\mu_H = \bar{\gamma} W_B w_Q \sigma_H \sigma_Q \rho \tag{31}$$

where $\bar{\gamma}$ is the wealth weighted harmonic average of the risk aversion of the two agents,

$$\bar{\gamma} = \sum_{j \in A,B} \left(\frac{W_j}{\gamma^j} \right)^{-1} \tag{32}$$

If for example ρ is positive, agent B will want to short the hedge asset, and agent A will take the other side of the trade since the asset is in zero net supply. The premium arises from the fact that Agent B is risk averse and wants to reduce the risk that comes from his untradable asset and will be willing to pay agent A to bear the risk of holding the hedge asset. It is possible to write the demands for each asset as a function of the exogenous variables and ρ^* once we insert the expressions for the equilibrium asset prices into equations (14) and (15). This yields

$$\begin{cases} w_M^{A*} = \frac{\bar{\gamma}}{\gamma^A} \\ w_H^{A*} = \frac{\bar{\gamma} W_B w_Q \sigma_Q \sigma_H \rho^*}{\gamma^A \sigma_H^2} \end{cases} \tag{33}$$

and for B,

$$\begin{cases} w_M^{B*} = \frac{\bar{\gamma}}{\gamma^B} \\ w_H^{B*} = \frac{W_B w_Q \sigma_Q \sigma_H \rho^*}{\sigma_H^2} \left(\frac{\bar{\gamma} W_B}{\gamma^B} - 1 \right) \end{cases} \tag{34}$$

From here we can also see how the demands of the hedging asset are affected by ρ. Agent A and Agent B will have demands with opposite signs since we can show the term in brackets is smaller than zero.

$$\frac{\bar{\gamma} W_B}{\gamma^B} - 1 < 0 \Leftrightarrow \frac{\gamma^A W_B}{\gamma^A W_B + \gamma^B W_A} < 1. \tag{35}$$

If for example ρ is positive, agent B will want to short the hedge asset, and agent A will take the other side of the trade since the asset is in zero net supply. The premium arises from the fact that Agent B is risk averse and wants to reduce the risk that comes from his untradable asset and will be willing to pay agent A to bear the risk of holding the hedge asset.

□

B Proof Proposition 3 and 4

Since the agent B's objective function is concave in w_M and w_H, and locally compact in the weights and ρ, then we can use the minimax theorem as long as the objective function is convex in ρ (Sion (1958)). Using this we can then re-write the problem of agent B from equation (6) as:

$$
\begin{aligned}
\min_{\{\rho\}} \max_{\{w_M^B, w_H^B\}} U^B = {} & r_f + w_M^B \mu_M + w_H^B \mu_H + w_Q \mu_Q \\
& - \frac{\gamma^B}{2} \left[(w_M^B)^2 \sigma_M^2 + (w_H^B)^2 \sigma_H^2 + (w_Q)^2 \sigma_Q^2 + 2 w_H^B w_Q \sigma_H \sigma_Q \rho \right] \quad (36) \\
s.t. \quad & \rho \leq \hat{\rho} + \eta \\
& \rho \geq \hat{\rho} - \eta
\end{aligned}
$$

From equation (8), we know the solution to the optimal hedge asset weights of agent B, which are expressed as a function of ρ. We then have to solve the minimization problem where we replace w_H^B for the optimal weight as a function of ρ. For notation purposes, let

$$
w_{H,\rho}^B = \frac{\mu_H}{\gamma^B \sigma_H^2} - \frac{w_Q \sigma_H \sigma_Q \rho}{\sigma_H^2}. \quad (37)
$$

Inserting this into the optimization problem, we now have,

$$
\begin{aligned}
\min_{\{\rho\}} U^B = {} & r_f + w_M^B \mu_M + w_{H,\rho}^B \mu_H + w_Q \mu_Q \\
& - \frac{\gamma^B}{2} \left[(w_M^B)^2 \sigma_M^2 + (w_{H,\rho}^B)^2 \sigma_H^2 + (w_Q)^2 \sigma_Q^2 + 2 w_{H,\rho}^B w_Q \sigma_H \sigma_Q \rho \right] \quad (38) \\
s.t. \quad & \rho \leq \hat{\rho} + \eta \quad (R1) \\
& \rho \geq \hat{\rho} - \eta \quad (R2)
\end{aligned}
$$

To simplify notation, the optimal weight of the hedge asset for agent B can be expressed as a function of ρ and denoted above by placing ρ in parenthesis. Also notice that we have also identified each restriction as R1 and R2. From the problem above we can then derive the Lagrangian function:

$$
\begin{aligned}
\mathcal{L} = \quad & r_f + w_M^B \mu_M + w_{H,\rho}^B \mu_H + w_Q \mu_Q \\
& - \frac{\gamma^B}{2} \left[(w_M^B)^2 \sigma_M^2 + (w_{H,\rho}^B)^2 \sigma_H^2 + (w_Q)^2 \sigma_Q^2 + 2 w_{H,\rho}^B w_Q \sigma_H \sigma_Q \rho \right] \\
& - \lambda_1 \left[\rho - \hat{\rho} - \eta \right] \\
& - \lambda_2 \left[\rho - \hat{\rho} + \eta \right]
\end{aligned} \quad (39)
$$

The optimality for this inequality constrained problem will then have the following necessary conditions:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \rho} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_1} \geq 0; \lambda_1 \geq 0; \lambda_1 . \frac{\partial \mathcal{L}}{\partial \lambda_1} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_2} \geq 0; \lambda_2 \geq 0; \lambda_2 . \frac{\partial \mathcal{L}}{\partial \lambda_2} = 0 \end{cases} \tag{40}$$

Solving the derivatives, the necessary conditions can be written as:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \rho} = \gamma^B w_Q^2 \sigma_Q^2 \rho - \frac{w_Q \sigma_H \sigma_Q \mu_H}{\sigma_H^2} + \lambda_1 - \lambda_2 = 0 \\ \rho \leq \hat{\rho} + \eta; \lambda_1 \geq 0; \lambda_1 . \frac{\partial \mathcal{L}}{\partial \lambda_1} = 0 \\ \rho \geq \hat{\rho} - \eta; \lambda_2 \geq 0; \lambda_2 . \frac{\partial \mathcal{L}}{\partial \lambda_2} = 0 \end{cases} \tag{41}$$

Furthermore, we can also show that the objective function of is globally convex in ρ given our initial assumptions since the second derivative of the utility function with respect to ρ is always positive.

$$\frac{\partial^2 U^B}{\partial \rho^2} = \gamma^B w_Q^2 \sigma_Q^2 > 0 \tag{42}$$

This means that agent B will pick the minimum ρ within the confidence interval, and ρ^* can either be an interior solution in which case the Lagrange multipliers λ_1 and λ_2 will both be zero, or a corner solution, where the Lagrange multiplier for the binding restriction will be negative and the other zero. Hence, we will have to solve the optimal demands of agent B and the equilibrium returns for 3 specific cases.

Case 1: No restriction is binding – $\lambda_1 = 0$ and $\lambda_2 = 0$.
Case 2: Restriction R1 is binding – $\lambda_1 > 0$ and $\lambda_2 = 0$.
Case 3: Restriction R2 is binding – $\lambda_1 = 0$ and $\lambda_2 > 0$.

Case 1

In this first case, no restriction is binding so both λ's are zero and we have to find the optimal ρ for agent B. Inserting this information to the necessary conditions in the equation system (41), this simplifies to:

$$\gamma^B w_Q^2 \sigma_Q^2 \rho - \frac{w_Q \sigma_H \sigma_Q \mu_H}{\sigma_H^2} = 0 \Leftrightarrow \rho = \frac{\mu_H}{\gamma^B w_Q \sigma_H \sigma_Q} \tag{43}$$

Once we insert this optimal ρ into the optimal demand of agent B in (8), we then find that the optimal weight for the hedge asset for agent B is zero.

$$w_H^B = 0 \tag{44}$$

If we insert this demand together with the demand of agent A from equation (7) into the market clearing condition (10), we obtain the equilibrium return for the case when both restrictions for ρ are non-binding.

$$\mu_H = 0 \tag{45}$$

Finally, we can insert the equilibrium return into (43) in order to find the optimal ρ for agent B in this case.

$$\rho = 0 \tag{46}$$

In terms of interpretation, this means that if agent B observes an estimate of ρ, $\hat{\rho}$, such that the confidence interval contains zero, then this will be them these results will characterize the equilibrium for this case.

Case 2

In this case, R1 is binding meaning that $\lambda_1 > 0$. Since both restrictions are mutually exclusive, this means that the second restriction will not be binding and therefore $\lambda_2 = 0$. If we input this information into the necessary conditions in equation (41), then we obtain the following:

$$\begin{cases} \gamma^B w_Q^2 \sigma_Q^2 \rho - \frac{w_Q \sigma_H \sigma_Q \mu_H}{\sigma_H^2} + \lambda_1 = 0 \\ \rho = \hat{\rho} + \eta \\ \lambda_1 > 0 \end{cases} \tag{47}$$

In this case, since R1 is binding, we know that the optimal ρ for agent B will be the upper bound of the confidence interval. If we insert this ρ into agent B's optimal demand for the hedging asset, we obtain:

$$w_H^B = \frac{\mu_H}{\gamma^B \sigma_H^2} - \frac{w_Q \sigma_H \sigma_Q (\hat{\rho} + \eta)}{\sigma_H^2} \tag{48}$$

Inserting the optimal demands of agent A and B, equations (7) and (48) respectively, into the market clearing conditions from equation (10), the the equilibrium return is given by:

$$\mu_H = \bar{\gamma} W^B w_Q \sigma_H \sigma_Q (\hat{\rho} + \eta) \tag{49}$$

where just as before, $\bar{\gamma}$ is the wealth weighted harmonic average of the risk aversion coefficients of agents A and B. We can see that in this case, the equilibrium return is higher that what it would be if agent B were to simply take the observed estimate of ρ as given. However we still do not know under which conditions this restriction is binding. For that we have to substitute the equilibrium return and the optimal ρ in this case into the necessary conditions for this case from equation (47).

$$\lambda_1 = \bar{\gamma} W^B w_Q^2 \sigma_Q^2 (\hat{\rho} + \eta) - \gamma^B w_Q^2 \sigma_Q^2 (\hat{\rho} + \eta) \tag{50}$$

Since $\lambda_1 > 0$, we must substitute the value of λ_1 above into this inequality.

$$\begin{aligned} &\bar{\gamma} W^B w_Q^2 \sigma_Q^2 (\hat{\rho} + \eta) - \gamma^B w_Q^2 \sigma_Q^2 (\hat{\rho} + \eta) > 0 \Leftrightarrow (\bar{\gamma} W^B - \gamma^B)(\hat{\rho} + \eta) > 0 \\ &\Leftrightarrow \hat{\rho} < -\eta \end{aligned} \tag{51}$$

From here we see that the upper bound of the confidence interval is the binding restriction whenever the observed estimate $\hat{\rho}$ is below zero and the confidence interval does not contain zero. Note that the last step in the equation above comes from the fact that $(\bar{\gamma} W^B - \gamma^B) < 0$ which we show in Appendix C in equation (35).

Case 3

For the third and final case, R2 is the binding restriction meaning that $\lambda_2 > 0$ and that $\lambda_1 = 0$. Applying this to the general necessary conditions from (41), we obtain:

$$\begin{cases} \gamma^B w_Q^2 \sigma_Q^2 \rho - \frac{w_Q \sigma_H \sigma_Q \mu_H}{\sigma_H^2} - \lambda_2 = 0 \\ \rho = \hat{\rho} - \eta \\ \lambda_2 > 0 \end{cases} \tag{52}$$

Inserting the optimal ρ from above into the hedge asset demand of B from (8), we get:

$$w_H^B = \frac{\mu_H}{\gamma^B \sigma_H^2} - \frac{w_Q \sigma_H \sigma_Q (\hat{\rho} - \eta)}{\sigma_H^2} \tag{53}$$

Inserting the optimal demands of agent A and B, equations (7) and (53) respectively, into the market clearing conditions from equation (10), the the equilibrium return is given by:

$$\mu_H = \bar{\gamma} W^B w_Q \sigma_H \sigma_Q (\hat{\rho} - \eta) \tag{54}$$

where just as before, $\bar{\gamma}$ is the wealth weighted harmonic average of the risk aversion coefficients of agents A and B. We can see that in this case, the equilibrium return is lower that what it would be if agent B were to simply take the observed estimate of ρ as given. However we still do not know under which conditions this restriction is binding. For that we have to substitute the equilibrium return and the optimal ρ in this case into the necessary conditions for this case from equation (52).

$$\lambda_2 = \gamma^B w_Q^2 \sigma_Q^2 (\hat{\rho} - \eta) - \bar{\gamma} W^B w_Q^2 \sigma_Q^2 (\hat{\rho} - \eta) \tag{55}$$

Since $\lambda_2 > 0$, we must substitute the value of λ_2 above into this inequality.

$$\begin{aligned} &\gamma^B w_Q^2 \sigma_Q^2 (\hat{\rho} - \eta) - \bar{\gamma} W^B w_Q^2 \sigma_Q^2 (\hat{\rho} - \eta) > 0 \Leftrightarrow (\gamma^B - \bar{\gamma} W^B)(\hat{\rho} - \eta) > 0 \\ &\Leftrightarrow \hat{\rho} > \eta \end{aligned} \tag{56}$$

From here we see that the lower bound of the confidence interval is the binding restriction whenever the observed estimate $\hat{\rho}$ is above zero and the confidence interval does not contain zero.

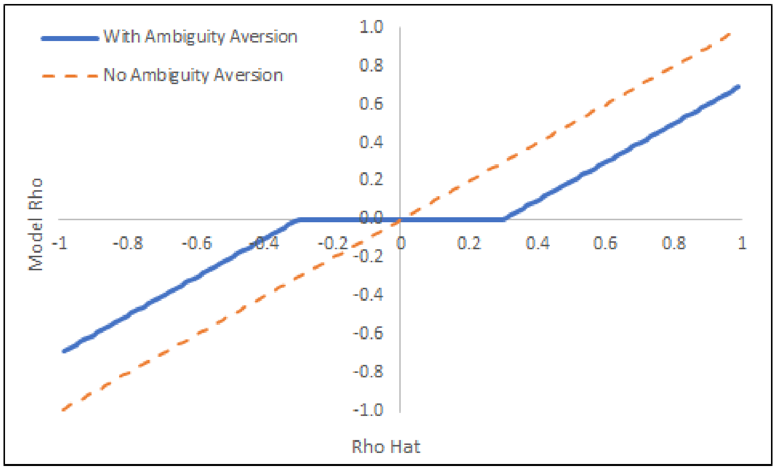
Since U^B is globally convex in ρ, it means the optimal ρ these solutions solve agent B's minimization problem. Once we incorporate cases 1 through 3, we obtain the result from proposition 2.

Figure 1: Comparing models with and without ambiguity aversion.

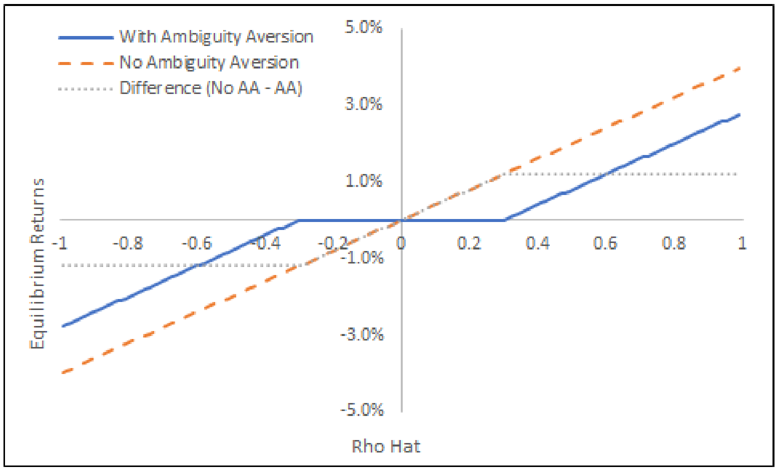
Panel A: Model Calibration. For the model with no ambiguity aversion, U is set to zero.

Variable	Value
γ^A	4
γ^B	4
σ_H	0.2
σ_Q	0.2
w_Q	0.5
W^A	0.5
W^B	0.5
η	0.3

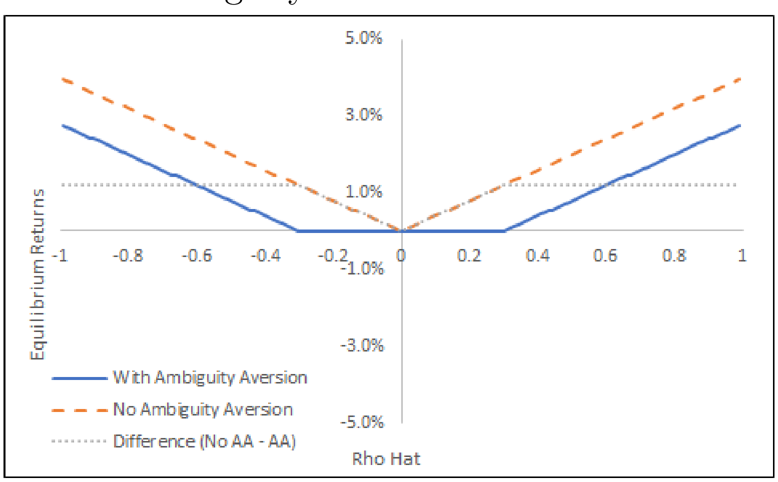
Panel B: Comparing ρ agent B uses to determine optimal demand.



Panel C: Comparing equilibrium μ_H under ambiguity aversion and no ambiguity aversion



Panel D: Comparing equilibrium μ_H in absolute terms under ambiguity aversion and no ambiguity aversion.



Tables

Table 1 – Predictive Regression Summary

Description: Cross sectional predictive regressions for beta risk measures. Every month we run a predictive regression using cross sectional data from the past 12 months in order to estimate the beta measure of the following 12 months. This table reports the average coefficient for each explanatory variable over the 624 months from January 1964 until December 2016. T-statistics are calculated using Fama-MacBeth standard errors, with 12 Newey-West lags due to the overlapping beta data. Independent variables were winsorized at the 2.5th and 97.5th percentiles and standardized so that the coefficients are interpreted as increases in the risk loading per standard deviation increase of the independent variable.

Panel A: Forecasted Risk Loading: Beta

Model		Beta	IV	Coskew	log (Size)	BTM	ROE	Leverage	Industry Dummies (Y/N)	Average Adj. R^2	RMSE
(1)	Avg. Coefficient	0.31							N	47.7%	0.34
	t-stat	21.44									
(2)	Avg. Coefficient	0.27	0.08						N	50.5%	0.33
	t-stat	20.55	11.87								
(3)	Avg. Coefficient	0.26	0.08	-0.01					N	50.9%	0.33
	t-stat	21.08	12.29	-2.14							
(4)	Avg. Coefficient	0.25	0.10	-0.01	0.03	-0.02	-0.09	-0.01	N	54.2%	0.33
	t-stat	18.67	16.39	-2.41	5.66	-3.24	-2.75	-1.09			
(5)	Avg. Coefficient	0.22	0.08	-0.01	0.02	-0.03	-0.10	0.02	Y	60.1%	0.32
	t-stat	16.21	19.12	-2.04	4.43	-5.19	-3.65	4.12			

Panel B: Forecasted Risk Loading: Beta DR

Model		Beta DR	IV	Coskew	log (Size)	BTM	ROE	Leverage	Industry Dummies (Y/N)	Average Adj. R^2	RMSE
(1)	Avg. Coefficient	-0.01							N	2.9%	0.31
	t-stat	-4.31									
(2)	Avg. Coefficient	-0.03	0.03						N	5.0%	0.31
	t-stat	-9.02	4.71								
(3)	Avg. Coefficient	-0.03	0.03	-0.02					N	5.8%	0.31
	t-stat	-8.93	4.75	-6.52							
(4)	Avg. Coefficient	-0.02	0.02	-0.01	-0.02	-0.02	0.01	0.02	N	9.0%	0.31
	t-stat	-6.79	3.41	-6.11	-4.57	-3.58	0.45	5.83			
(5)	Avg. Coefficient	-0.02	0.01	-0.01	-0.03	-0.01	0.01	0.03	Y	13.8%	0.30
	t-stat	-6.86	3.05	-5.31	-5.31	-3.83	0.68	7.29			

Panel C: Forecasted Risk Loading: Beta SMB

Model		Beta SMB	IV	Coskew	log (Size)	BTM	ROE	Leverage	Industry Dummies (Y/N)	Average Adj. R^2	RMSE
(1)	Avg. Coefficient	0.11							N	9.5%	0.52
	t-stat	6.01									
(2)	Avg. Coefficient	-0.02	0.20						N	18.9%	0.50
	t-stat	-1.18	15.26								
(3)	Avg. Coefficient	-0.02	0.19	-0.01					N	20.0%	0.50
	t-stat	-0.95	14.95	-2.33							
(4)	Avg. Coefficient	0.04	0.10	0.00	-0.18	-0.02	-0.17	-0.03	N	31.5%	0.47
	t-stat	2.95	9.63	-0.87	-20.97	-2.68	-2.52	-3.75			
(5)	Avg. Coefficient	0.02	0.09	0.00	-0.18	-0.02	-0.14	0.00	Y	37.2%	0.46
	t-stat	1.49	10.25	-1.44	-22.98	-2.04	-2.61	-0.33			

Panel D: Forecasted Risk Loading: Beta HML

Model		Beta HML	IV	Coskew	log(Size)	BTM	ROE	Leverage	Industry Dummies (Y/N)	Average Adj. R^2	RMSE
(1)	Avg. Coefficient	-0.42							N	14.5%	0.77
	t-stat	-7.98									
(2)	Avg. Coefficient	-0.32	-0.10						N	18.1%	0.76
	t-stat	-7.19	-3.97								
(3)	Avg. Coefficient	-0.32	-0.10	-0.03					N	19.7%	0.76
	t-stat	-7.10	-4.07	-2.48							
(4)	Avg. Coefficient	-0.22	-0.08	-0.02	-0.01	0.17	0.14	0.13	N	35.9%	0.69
	t-stat	-5.70	-4.03	-1.77	-1.58	11.82	1.62	9.64			
(5)	Avg. Coefficient	-0.18	-0.07	-0.01	-0.01	0.16	0.06	0.10	Y	45.6%	0.66
	t-stat	-6.63	-4.21	-1.40	-1.73	13.95	0.88	9.67			

Table 2 – Beta portfolio sorts

Description: This table summarizes the value weighted average returns on portfolios formed on realized, predicted and pre-ranking beta measures described in the panels. Each month, Betas are calculated using daily returns of the past 12 months. These betas are then used to sort stocks and form value weighted quintile portfolios invested in the same time period in which the betas were estimated for the realized betas and invested over the next 12 months for the predicted and pre-ranking betas. Over the sample, the number of stocks in each portfolio ranges from 17 to 181 stocks. Returns are in excess of the 1 month treasury bill over the same 12 month period. Market Beta, Beta DR, Beta SMB and Beta HML are the value weighted average realized beta loadings in the portfolios over time. The Hi-Lo portfolio is the long short portfolio going long the high and shorting the low beta portfolios, and the t-stat for raw returns and betas is calculated using Newey-West heteroskedastic-robust standard errors with 12 lags. CAPM Alphas is calculated using a CAPM regression with Newey-West heteroskedastic-robust standard errors with 12 lags. The sample period ranges from January 1965 until December 2016 and uses monthly observations.

Panel A - Relative Downside Beta Sorts

Realized Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Beta DR
1 Lo Beta	2.69%	-4.24%	-0.41
2	5.70%	0.01%	-0.14
3	7.55%	2.18%	0.00
4	8.36%	2.82%	0.15
5 Hi Beta	8.94%	3.20%	0.41
Hi-Lo	6.25%	7.45%	0.81
t-stat	4.820	4.991	

Predicted Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Beta DR
1 Lo Beta	5.53%	-0.46%	-0.07
2	6.21%	1.26%	-0.01
3	6.91%	1.82%	0.01
4	8.08%	2.89%	0.02
5 Hi Beta	9.66%	3.92%	0.05
Hi-Lo	4.13%	4.37%	0.12
t-stat	2.632	2.588	8.083

Pre Ranking Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Beta DR
1 Lo Beta	6.21%	-0.64%	-0.08
2	6.61%	1.15%	-0.04
3	6.68%	1.27%	-0.03
4	6.36%	1.15%	-0.01
5 Hi Beta	6.42%	0.58%	0.02
Hi-Lo	0.21%	1.22%	0.10
t-stat	0.142	0.931	7.124

Panel B - SMB Beta Sorts

Realized Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Beta SMB
1 Lo Beta	6.26%	0.99%	-0.59
2	7.22%	2.20%	-0.17
3	6.95%	1.25%	0.09
4	6.41%	-0.30%	0.36
5 Hi Beta	4.66%	-3.22%	0.89
Hi-Lo	-1.60%	-4.21%	1.48
t-stat	-0.673	-1.821	

Predicted Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Beta SMB
1 Lo Beta	5.37%	0.02%	-0.33
2	6.91%	1.01%	-0.03
3	7.63%	1.52%	0.11
4	8.72%	2.28%	0.22
5 Hi Beta	10.79%	3.30%	0.35
Hi-Lo	5.42%	3.28%	0.67
t-stat	2.569	1.764	26.576

Pre Ranking Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Beta SMB
1 Lo Beta	5.59%	0.60%	-0.37
2	6.57%	1.28%	-0.18
3	7.61%	2.04%	-0.04
4	7.65%	1.10%	0.10
5 Hi Beta	7.55%	0.08%	0.30
Hi-Lo	1.96%	-0.52%	0.66
t-stat	0.999	-0.257	26.367

Panel C - HML Beta Sorts

Realized Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Beta HML
1 Lo Beta	7.43%	0.42%	-1.09
2	6.87%	0.82%	-0.32
3	6.51%	1.49%	0.06
4	6.34%	1.21%	0.39
5 Hi Beta	6.11%	0.75%	1.01
Hi-Lo	-1.32%	0.33%	2.09
t-stat	-0.487	0.116	

Predicted Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Beta HML
1 Lo Beta	6.20%	-0.69%	-0.58
2	5.48%	-0.06%	-0.15
3	6.88%	1.50%	0.11
4	7.42%	2.71%	0.32
5 Hi Beta	7.97%	3.79%	0.55
Hi-Lo	1.76%	4.49%	1.13
t-stat	0.789	2.001	19.078

Pre Ranking Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Beta HML
1 Lo Beta	4.97%	-1.79%	-0.62
2	6.55%	0.65%	-0.17
3	7.11%	1.70%	0.04
4	7.68%	2.53%	0.21
5 Hi Beta	7.15%	2.34%	0.52
Hi-Lo	2.19%	4.13%	1.13
t-stat	0.958	1.772	17.267

Panel D - Market Beta Sorts

Realized Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Market Beta
1 Lo Beta	4.68%	1.82%	0.45
2	5.11%	0.77%	0.75
3	6.11%	0.66%	0.98
4	7.23%	0.24%	1.22
5 Hi Beta	10.00%	0.49%	1.64
Hi-Lo	5.32%	-1.33%	1.19
t-stat	1.648	-0.533	

Predicted Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Market Beta
1 Lo Beta	5.95%	2.49%	0.56
2	7.15%	2.47%	0.82
3	6.16%	0.82%	0.93
4	6.02%	-0.20%	1.06
5 Hi Beta	6.83%	-1.14%	1.27
Hi-Lo	0.87%	-3.64%	0.71
t-stat	0.358	-1.646	18.900

Pre Ranking Beta Portfolio Sorts

Portfolio	Raw Return	CAPM Alpha	Market Beta
1 Lo Beta	5.82%	2.40%	0.57
2	6.86%	2.05%	0.81
3	6.70%	1.18%	0.95
4	7.10%	0.71%	1.10
5 Hi Beta	5.48%	-2.36%	1.33
Hi-Lo	-0.34%	-4.75%	0.77
t-stat	-0.140	-2.142	20.713

Table 3 – Hedge Portfolio Beta Distributions

Description: This table looks at the distributions of the realized time-series betas of the hedge portfolios from the portfolio sorts. Each of these beta distributions corresponds to the beta used for the portfolio sort, i.e. the Beta DR row is the Beta DR of the Long-Short Portfolio formed on either pre ranking or predicted Beta DRs. Panel A shows the distribution of realized betas for the long-short portfolios formed on pre-ranking betas, while Panel B shows the distribution of realized betas of the long-short portfolios sorted on predicted betas.

Panel A - Pre Ranking Beta

Confidence Interval		99%		95%		90%	
	Mean	LL	UL	LL	UL	LL	UL
Market Beta	0.77	0.22	2.12	0.34	1.61	0.39	1.19
Beta DR	0.10	-0.16	0.43	-0.10	0.36	-0.08	0.33
Beta SMB	0.66	0.20	1.11	0.28	1.01	0.32	0.97
Beta HML	1.13	-0.44	2.82	0.14	2.20	0.28	2.00

Panel B - Predicted Beta

Confidence Interval		99%		95%		90%	
	Mean	LL	UL	LL	UL	LL	UL
Market Beta	0.71	0.16	1.93	0.29	1.65	0.35	1.19
Beta DR	0.12	-0.23	0.39	-0.14	0.34	-0.09	0.31
Beta SMB	0.67	-0.01	1.10	0.30	1.02	0.35	0.96
Beta HML	1.13	0.06	2.80	0.18	2.17	0.30	1.78

Table 4 – Stock Summary Statistics

Description: Summary statistics for stock betas and controls used in the Fama MacBeth regressions. In Panel A, the mean is the time series mean of cross sectional means of the variable and the standard deviation is the time series mean of the cross sectional standard deviations of each variable. The percentile and median statistics are the time series mean of the cross sectional percentiles and medians. In Panel B, the mean is the cross sectional mean of the time series means of each stock. The standard deviation is the cross sectional mean of the time series standard deviations of each stock. The average autocorrelation, is the mean of the 12 month autocorrelation of the betas for each stock. Note that given the 12 months used to estimate betas, we are measuring the autocorrelations of betas estimated without overlapping windows.

Panel A - Cross Section

	Mean	Standard Deviation	99th Percentile	Median	1st Percentile
Realized Betas					
Market Beta	1.014	0.459	0.199	0.966	2.265
Beta D	1.018	0.538	-0.004	0.957	2.530
Beta DR	0.004	0.315	-0.799	0.003	0.816
Beta SMB	0.131	0.538	-0.975	0.087	1.635
Beta HML	-0.004	0.782	-2.174	0.056	1.763
Predicted Betas					
Market Beta	0.977	0.281	0.427	0.974	1.620
Beta D	0.986	0.286	0.432	0.977	1.657
Beta DR	-0.003	0.062	-0.169	0.000	0.129
Beta SMB	0.078	0.224	-0.463	0.100	0.537
Beta HML	0.021	0.363	-0.787	0.022	0.804
Controls					
Idiosyncratic Vol	0.017	0.006	0.008	0.016	0.035
Coskew	-0.112	0.150	-0.456	-0.113	0.236
logSize	7.641	1.008	6.175	7.433	10.736
BTM	0.600	0.378	0.080	0.532	1.835

Panel B - Time Series

	Mean	Standard Deviation	Average Autocorrelation
Realized Betas			
Market Beta	1.129	0.338	0.205
Beta D	1.144	0.454	0.104
Beta DR	0.015	0.318	-0.049
Beta SMB	0.361	0.459	0.032
Beta HML	-0.202	0.656	0.098

Table 5 – Fama MacBeth Regressions

Summarizing results from equation (XX): $R_{it} = \lambda_{0,t} + \lambda_{1,t} E_{t-1}[\beta_{it}^{R}] + \delta_{it}$, we present the average risk premium estimates and their respective t-stats using Fama-MacBeth regressions with 12 Newey West Lags. Each column has a different β as an independent variable. Panel A uses realized betas to explain the cross-section of returns, while Panel B uses pre ranking betas. Controls in both these panels are the values available at the beginning of the year in which we evaluate returns on the left hand side. Panel C uses direct beta forecasts estimated at t-1 estimated from the best model from Table 1 (Model 5). All variables are winsorized at the 1st and 99th percentiles and all right hand side variables are standardized every period so that we can more easily compare across different betas. We do not use controls in Panel C due to multicollinearity issues since predicted betas are a linear combination of some of the control variables.

Panel A - Contemporaneous Betas

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Market Beta		0.023	0.017		0.023	-0.016		0.021	-0.018	0.024	0.032
t-stat		1.890	5.289		2.036	-2.405		1.900	-2.378	2.235	2.885
Beta DR	0.024	0.019	0.033							0.014	0.012
t-stat	5.087	5.571	2.609							5.482	5.234
Beta SMB				-0.003	-0.006	0.032				-0.010	-0.020
t-stat				-0.341	-0.902	2.824				-1.305	-2.840
Beta HML							-0.009	-0.006	0.029	-0.008	-0.022
t-stat							-0.868	-0.756	2.491	-1.087	-3.165
iVol			-0.011			-0.006			-0.012		-0.009
t-stat			-2.297			-1.200			-2.621		-2.118
Coskew			0.000			-0.001			0.000		0.000
t-stat			-0.014			-0.271			-0.196		-0.140
Size			-0.016			-0.023			-0.017		-0.024
t-stat			-3.724			-6.329			-4.144		-7.046
BTM			0.012			0.010			0.019		0.018
t-stat			2.501			2.161			4.332		4.375
Intercept	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Average Adj. R^2	2.0%	10.4%	15.5%	4.1%	11.3%	16.1%	5.7%	12.9%	17.3%	16.0%	19.7%
Observations	309199	309199	309199	309199	309199	309199	309199	309199	309199	309199	309199

Panel B - Pre Ranking Betas

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
Beta		-0.004	-0.002		-0.002	0.005		-0.002	-0.001	-0.003	-0.005
t-stat		-0.412	-0.730		-0.305	1.307		-0.211	-0.108	-0.396	-0.759
Beta DR	0.000	-0.001	-0.005							-0.001	-0.002
t-stat	0.116	-0.369	-0.781							-0.410	-0.888
Beta SMB				0.004	0.005	-0.002				0.002	0.002
t-stat				0.612	1.061	-0.366				0.556	0.512
Beta HML							0.009	0.006	-0.003	0.000	-0.004
t-stat							1.146	1.010	-0.538	0.070	-0.827
iVol			0.000			-0.001			-0.001		-0.001
t-stat			0.047			-0.173			-0.235		-0.129
Coskew			-0.002			-0.001			-0.002		-0.003
t-stat			-1.059			-0.262			-0.689		-1.201
Size			-0.001			0.001			-0.001		0.000
t-stat			-0.244			0.221			-0.156		0.047
BTM			0.006			0.006			0.006		0.006
t-stat			1.221			1.195			1.422		1.461
Intercept	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Average Adj. R^2	0.6%	5.3%	9.7%	2.1%	5.9%	9.9%	3.0%	6.5%	10.3%	7.9%	11.1%
Observations	335867	335867	335867	335867	335867	335867	335867	335867	335867	335867	335867

Panel C - Predicted Betas

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Market Beta		0.026		0.015		0.032	0.027
t-stat		2.118		1.197		2.423	2.094
Beta DR	0.016	0.019					0.011
t-stat	2.528	4.446					1.882
Beta SMB			0.022	0.019			0.009
t-stat			3.406	3.542			1.224
Beta HML					0.006	0.018	0.018
t-stat					0.720	2.998	3.060
Intercept	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Average Adj. R^2	2.1%	10.2%	2.9%	11.0%	4.3%	11.7%	14.7%
Observations	309110	309110	309110	309110	309110	309110	309110

Table 6 – Optimal portfolio weights

Description: In this table we look at the optimal weights for a CRRA investor estimated from equation (XXX). As hedge portfolios, we use the long-short portfolios from Tables 3 to 5, formed by going long the highest quintile of beta estimates and short the lowest quintile. Every month, we form portfolios and then hold them for a year. Due to the overlapping returns, we correct t-stats using Newey-West standard errors with 12 lags.

Panel A: Only Market Portfolio

	Gamma				
	1	**2**	**5**	**10**	**20**
Market Weight	1.72	0.98	0.42	0.22	0.12
t-stat	3.78	2.90	2.73	2.68	2.54

Panel B: Contemporaneous Betas

Relative downside beta hedge portfolio

	Gamma				
	1	**2**	**5**	**10**	**20**
Market Weight	1.75	1.26	0.59	0.32	0.18
t-stat	4.11	4.11	3.86	3.89	3.62
Hedge Weight	3.58	2.61	1.18	0.62	0.33
t-stat	8.61	5.90	5.42	5.83	5.82

SMB beta hedge portfolio

	Gamma				
	1	**2**	**5**	**10**	**20**
Market Weight	1.94	1.15	0.50	0.27	0.15
t-stat	6.48	3.33	3.10	3.06	2.89
Hedge Weight	-1.22	-0.46	-0.20	-0.11	-0.06
t-stat	-30.58	-1.85	-1.76	-1.80	-1.92

HML beta hedge portfolio

	Gamma				
	1	**2**	**5**	**10**	**20**
Market Weight	1.73	0.98	0.42	0.22	0.12
t-stat	3.70	2.82	2.65	2.60	2.48
Hedge Weight	0.05	0.02	0.00	-0.01	-0.01
t-stat	0.10	0.07	-0.04	-0.20	-0.45

Panel C: Pre Ranking Betas

Relative downside beta hedge portfolio

	Gamma				
	1	**2**	**5**	**10**	**20**
Market Weight	1.74	1.00	0.43	0.23	0.12
t-stat	4.00	2.97	2.76	2.69	2.51
Hedge Weight	0.52	0.30	0.12	0.06	0.03
t-stat	0.66	0.67	0.65	0.59	0.49

SMB beta hedge portfolio

	Gamma				
	1	**2**	**5**	**10**	**20**
Market Weight	1.77	1.02	0.44	0.23	0.13
t-stat	3.51	2.75	2.62	2.59	2.42
Hedge Weight	-0.16	-0.11	-0.05	-0.03	-0.02
t-stat	-0.24	-0.28	-0.34	-0.39	-0.40

HML beta hedge portfolio

	Gamma				
	1	**2**	**5**	**10**	**20**
Market Weight	1.98	1.17	0.50	0.25	0.13
t-stat	5.63	3.33	2.99	2.85	2.60
Hedge Weight	1.18	0.63	0.24	0.11	0.05
t-stat	1.88	1.82	1.72	1.59	1.34

Panel D: Predicted Betas

Relative downside beta hedge portfolio

	Gamma				
	1	**2**	**5**	**10**	**20**
Market Weight	1.78	1.04	0.45	0.24	0.13
t-stat	4.31	3.06	2.83	2.78	2.60
Hedge Weight	2.54	1.43	0.59	0.30	0.16
t-stat	3.71	2.71	2.55	2.58	2.69

SMB beta hedge portfolio

	Gamma				
	1	**2**	**5**	**10**	**20**
Market Weight	1.25	0.74	0.33	0.18	0.10
t-stat	2.23	1.99	1.96	1.96	1.91
Hedge Weight	1.70	0.85	0.32	0.15	0.07
t-stat	2.20	1.97	1.87	1.76	1.46

HML beta hedge portfolio

	Gamma				
	1	**2**	**5**	**10**	**20**
Market Weight	1.94	1.21	0.52	0.26	0.13
t-stat	6.98	3.34	2.92	2.74	2.47
Hedge Weight	1.23	0.72	0.28	0.13	0.05
t-stat	1.88	1.85	1.73	1.54	1.19

Panel E: Predicted Betas and Fama French SMB and HML as Control Portfolios

Relative downside beta hedge portfolio

	Gamma				
	1	**2**	**5**	**10**	**20**
Market					
Weight	1.45	1.11	0.51	0.26	0.13
t-stat	3.98	3.35	2.67	2.46	2.26
Hedge					
Weight	1.88	1.00	0.40	0.20	0.12
t-stat	1.85	1.57	1.49	1.53	1.70
SMB					
Weight	0.05	-0.01	0.00	0.01	0.02
t-stat	0.05	-0.01	0.00	0.08	0.20
HML					
Weight	2.36	1.62	0.67	0.31	0.11
t-stat	2.53	3.01	2.76	2.46	1.76