CFR Working Paper NO. 21-07

Multivariate Crash Risk

F. Chabi-Yo • M. Huggenberger • F. Weigert

centre for financial research
cologne

Multivariate Crash Risk*

Fousseni Chabi-Yo[a], Markus Huggenberger[b], Florian Weigert[c]

This version: May 21, 2021

Abstract: This paper investigates whether multivariate crash risk (MCRASH), defined as exposure to extreme realizations of multiple systematic factors, is priced in the cross-section of expected stock returns. We derive an extended linear model with a positive premium for MCRASH and we empirically confirm that stocks with high MCRASH earn significantly higher future returns than stocks with low MCRASH. The premium is not explained by linear factor exposures, alternative downside risk measures or stock characteristics. Extending market-based definitions of crash risk to other well-established factors helps to determine the cross-section of expected stock returns without further expanding the factor zoo.

Keywords: Asset pricing, Non-linear dependence, Crash aversion, Downside risk, Tail risk, Lower tail dependence, Copulas

JEL classifications: C58, G01, G11, G12, G17.

---

* We are grateful for helpful comments and suggestions from Turan G. Bali, Matthias Fengler, Karl Frauendorfer, Junye Li, Marcel Prokopczuk, Ernst Maug, Stefan Ruenzi, Bruno Solnik, Gregor Weiß, Lai Xu and from seminar participants at the University of Mannheim, the 2018 Annual Meeting of the German Finance Association, the 2019 Annual Society for Financial Econometrics Conference (SoFiE), the 2019 International Risk Management Conference (IRMC), the 2019 China International Conference in Finance (CICF) and the 2021 SoFiE Seminar Series. The authors acknowledge support by the state of Baden-Württemberg through bwHPC and the German Research Foundation (DFG) through grant INST 35/1134-1 FUGG.

[a] University of Massachusetts in Amherst, 307B, Isenberg, 121 Presidents Drive, Amherst, MA, 01003. Email: fchabiyo@umass.edu.

[b] University of Mannheim, Schloss, 68131 Mannheim, Germany. Email: markus.huggenberger@uni-mannheim.de.

[c] University of Neuchâtel, Institute of Financial Analysis, Faculty of Economics and Business, Rue A.-L. Breguet 2, 2000 Neuchâtel, Switzerland. Email: florian.weigert@unine.ch.

1 Introduction

The relationship between left tail risk and the cross-section of expected stock returns has received considerable attention in the recent empirical asset pricing literature (Ang et al., 2006a; Kelly and Jiang, 2014; van Oordt and Zhou, 2016; Chabi-Yo et al., 2018; Jang and Kang, 2019; Lu and Murray, 2019; Atilgan et al., 2020). So far, these studies either focus on a stock's univariate crash risk (e.g., its crash probability, Value-at-Risk, or Expected Shortfall) or its bivariate crash risk with the market (e.g., its downside beta, tail beta, lower tail dependence, or option-implied bear beta). To the best of our knowledge, a stock's sensitivity to market crashes *and* extreme downside realizations of additional risk factors has not been examined yet. In this paper, we fill this gap and investigate the relationship between multivariate crash risk and the cross-section of average stock returns.

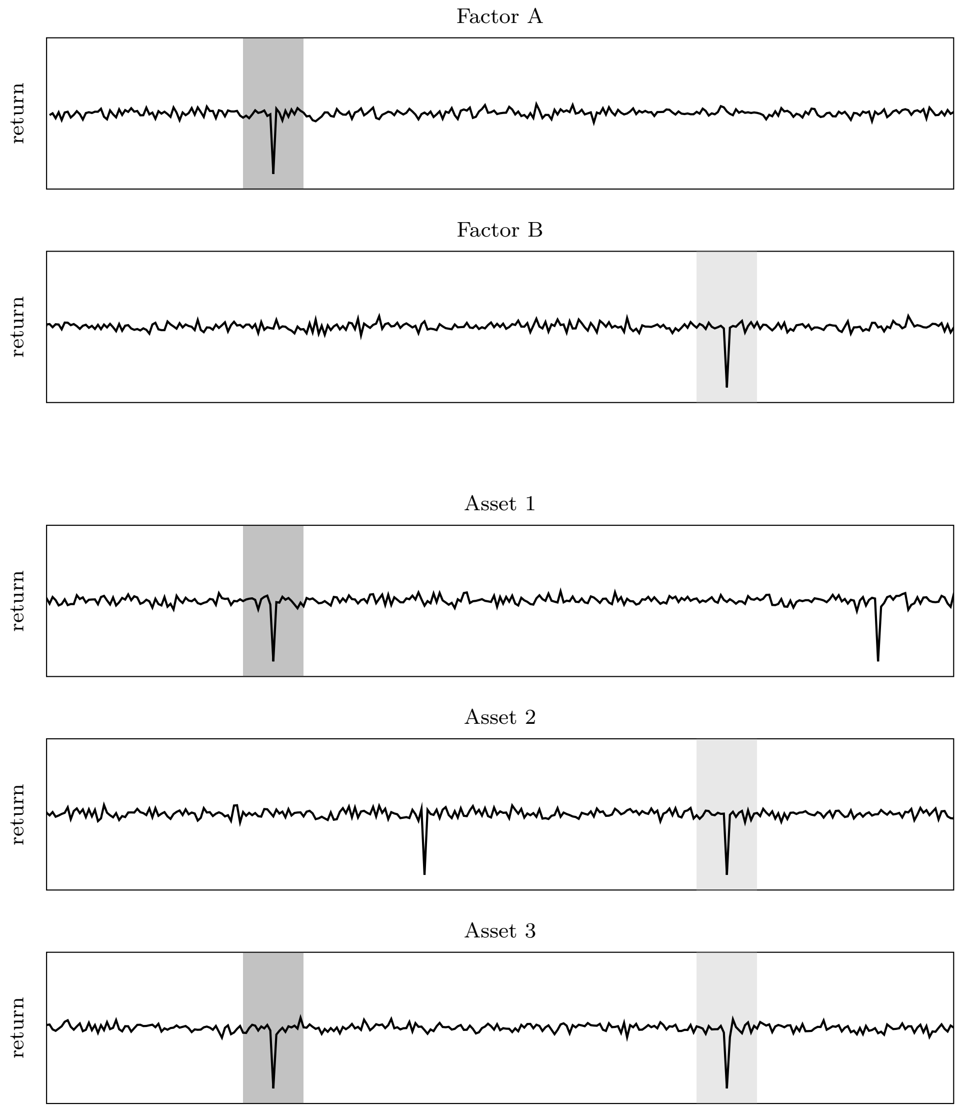
Since the seminal paper of Fama and French (1993), it is well accepted that the stochastic discount factor (SDF) cannot be spanned by the market alone, but that it depends on additional (non-market) risk factors. Nevertheless, bivariate crash risk measures proposed in the literature exclusively use the market factor to quantify a stock's non-linear exposure to adverse scenarios. By construction, such market-based measures neglect tail risk exposure that is driven by extreme realizations of other priced factors and they only partially capture exposure to extreme states driven by joint tail events of several factors. If such left tail realizations of non-market factors are important for the wealth of a representative investor, then an asset's exposure to these crash events should help explain the cross-section of expected stock returns.

To capture a stock's sensitivity to crash events for a set of priced factors, we propose a new systematic risk measure that we refer to as MCRASH. We define MCRASH of a stock as the conditional probability that the stock realizes a left tail event given that at least one of the risk factors realizes a left tail event at the same point in time. We propose to use quantile-based thresholds for the corresponding left tail events of the stock and the factors, which implies that MCRASH is not influenced by univariate risk characteristics. With this definition, MCRASH can be seen as a generalization of bivariate lower tail dependence coefficients (see e.g. Poon et al., 2004). More specifically, for the case of the market as a single factor, MCRASH boils down to the bivariate lower tail dependence between a stock and the market as used in Chabi-Yo et al. (2018).

To obtain a first intuition why MCRASH is distinct from a stock's bivariate crash risk with the market, we provide a stylized example in Figure 1. It shows the return time series of two risk factors (e.g., A = market, B = non-market) and three individual assets (1, 2, and 3). The individual (univariate) crash risk of the three assets is comparable as each of the assets realizes two large negative returns of similar magnitude. However, the three assets differ with respect to their sensitivity to the crashes of the two systematic factors. Asset 1 realizes a simultaneous crash with factor A but it is not negatively affected by the crash of factor B. Conversely, asset 2 realizes a joint left tail event with factor B but not with factor A. Finally, asset 3 realizes simultaneous crashes with both systematic factors. A bivariate crash risk measure concentrating solely on risk factor A (i.e., the market) would indicate that the crash risk exposure of asset 1 and 3 is identical and that the systematic crash risk of asset 2 is zero. In contrast, MCRASH accounts for the crash risk exposure of asset 2 and assigns the highest level of systematic crash risk to asset 3, which realizes simultaneous crashes with both risk factors A and B. If an investor cares about tail events of both factors, she should require a higher crash-related premium for holding asset 3 than for asset 1 or 2.

To formalize this idea, we analyze the relevance of multivariate crash risk for asset prices in a setting, where the true unknown SDF is replaced by its projection on a given set of factors. By definition, this projection can then be written as a (measurable) function of the factor returns and from applying a first-order Taylor series expansion, the well-known decomposition of an asset's risk premium in terms of linear factor betas is derived. We propose a simple extension of this standard

Figure 1: Stylized Example – Multivariate Crash Risk



This figure provides a stylized example for the concept of multivariate crash risk. The first two graphs show the return time series of two risk factors A and B. The following three panels show the time series of stocks whose returns are assumed to be driven by these factors. There is one large crash event for each of the two factors. The crash of factor A (B) and simultaneous crash events of the stocks are highlighted in dark (light) gray.

argument by adding a term that improves the approximation quality for left tail events. This extension allows us to link the risk premium to a stock's multivariate crash sensitivity as measured by MCRASH. In other words, we derive an extension of standard linear multifactor models, where MCRASH captures the premium for (non-linear) exposure to systematic crash risk. Specifically, our theoretical results imply that a stock's expected excess return is increasing in MCRASH if the projected SDF is a convex function of the factor returns.

To empirically verify this theoretical prediction, we employ daily return data for U.S. common stocks trading on the NYSE/AMEX/NASDAQ from 1964 to 2018 and estimate monthly MCRASH measures for each stock i and month t. In our main analysis, we measure MCRASH with respect to a seven factor model that adds a momentum factor and a low risk factor to the factors proposed by Fama and French (1993, 2015). In particular, we include the market factor (MKT), the SMB size factor, the HML value factor, the RMW profitability factor, the CMA investment factor, the UMD momentum factor as in Carhart (1997) and the BAB betting-against-beta factor as in Frazzini and Pedersen (2014).

In our baseline specification, MCRASH is computed at the 5%-probability level using a rolling window of 250 daily returns (i.e., one year of daily data). We apply a semiparametric methodology that combines parametric GARCH models for the marginal return distributions of the stock and the risk factors with a nonparametric estimation of the dependence structure. In this way, we account for volatility clustering, but do not impose a restrictive form on the (potentially non-linear) dependencies between stock and factor returns.[1] For the average cross-section, the resulting MCRASH estimates range between 0 and 0.17 with a mean of 0.08. The average cross-sectional correlation of MCRASH with linear factor betas and firm characteristics included in our empirical analysis is only moderate with a maximum value of 0.28 attained for the market beta.

[1]Christoffersen and Langlois (2013) provide evidence for extreme non-linear dependencies among the four Carhart (1997) factors, despite small or even negative linear correlations.

In our main asset pricing tests, we relate a stock's MCRASH estimated at the end of month t to future returns and alphas in the month $t+1$. Results from equal-weighted univariate portfolio sorts reveal that excess returns in month $t+1$ monotonically increase in the level of MCRASH at the end of month t. The return spread between stocks with the highest MCRASH (portfolio 10) and the lowest MCRASH (portfolio 1) amounts to annualized 4.68% and is statistically significant at the 1% level with a Newey and West (1987) t-statistic of 3.69. When the MCRASH (10)-(1) return spread is adjusted for the market as well as SMB, HML, RMW, CMA, UMD, and BAB, it amounts to annualized 5.28% and is statistically significant at the 1% level with a t-statistic of 4.79. Hence, accounting for linear exposure to the factors used in the estimation of MCRASH even increases the statistical and economic significance of the MCRASH (10)-(1) return spread. We also observe that the MCRASH (10)-(1) return spread remains statistically significant and economically large when we risk-adjust it by alternative factor models proposed in the literature. Moreover, the impact of MCRASH is not limited to one-month-ahead performance, but remains strong for risk-adjusted cumulative returns up to month $t+6$.

The positive pricing effect of MCRASH on the cross-section of average stock returns is confirmed in multivariate tests. Results from Fama and MacBeth (1973) regressions of excess returns in month $t+1$ on MCRASH in month t controlling for linear risk exposures (i.e., stock betas to different asset pricing factors) and firm characteristics (size, book-to-market, momentum, reversal, stock illiquidity, and maximum daily return in month t) indicate that MCRASH is a positive determinant of expected future stock returns. In a specification with all factor betas, matching the extended linear model from our theory, the coefficient estimate for MCRASH is 4.37 with a t-statistic of 5.89. Across specifications, the coefficient estimates range between 2.69 and 5.56 with t-statistics between 3.58 and 6.70. Given a (90% - 10%)-interquantile spread of 0.08 between stocks with the highest and lowest MCRASH, these coefficient estimates translate into annualized premiums between 2.58% and 5.34%.

The positive relationship between MCRASH and future returns holds when we control for other downside and tail risk measures proposed in the literature. Our results based on multivariate regressions and bivariate portfolio sorts reveal that the return effect of MCRASH remains robust and statistically significant when controlling for a stock's downside beta (Ang et al., 2006a), tail beta (Kelly and Jiang, 2014), idiosyncratic volatility and idiosyncratic skewness (Ang et al., 2006b), coskewness and cokurtosis (Harvey and Siddique, 2000), value-at-risk (Atilgan et al., 2020), and bear beta (Lu and Murray, 2019). These findings provide strong evidence that investors care about the multivariate crash risk of stocks and that the MCRASH return premium is not already subsumed in market-based downside and tail risk premiums.

We conduct different robustness checks to confirm the significantly positive association between MCRASH and future returns. Our results are stable when we specify different filters for our stock sample, alter the threshold for left tail events, perform a sample split and examine the time periods from 1965 to 1991 and 1992 to 2018 separately, apply different methodologies in the estimation of MCRASH, and use the 49 value-weighted Fama and French industry portfolios as test assets. We also find a significant relation between MCRASH and risk-adjusted future returns when we perform value-weighted portfolio sorts and exclude the top 1% largest stocks in the cross-section.[2] Finally, we document that the premium captured by MCRASH is not specific to our choice of the seven factors used in the baseline analysis. When we estimate MCRASH for common subsets of our seven factors and the factor models proposed by Stambaugh and Yuan (2017) and Hou et al. (2021), we continue to find a positive and significant relationship between the model-specific MCRASH measures and future returns suggesting that a non-linear crash risk premium can help to improve many standard pricing models.

[2] In value-weighted portfolio sorts with all stocks in the cross-section, the risk-adjusted MCRASH (10)-(1) return spread amounts to annualized 1.92% with a t-statistic of 1.39. Hence, MCRASH has a significant effect on future returns for all stocks *except* for the largest 1% in the cross-section.

We finally investigate the pricing of alternative notions of multivariate crash risk. First, we calculate *bivariate* crash risk measures for the market and each of the six non-market factors used for the computation of MCRASH.[3] In line with the results of Chabi-Yo et al. (2018), we observe that a stock's sensitivity to market crashes carries a positive risk premium in the cross-section of future stock returns; moreover, our results reveal a weakly significant premium for exposure to crashes of the size factor and the profitability factor. However, multivariate regressions and dependent double sorts indicate that the effect of MCRASH on future returns is not subsumed by any of the bivariate crash risk measures. To the contrary, the return effects of bivariate crash measures for the market, the size, and the profitability factor become insignificant when we control for the impact of MCRASH (and firm characteristics) in multivariate regressions.

Second, we turn our focus to *joint* crashes of multiple factors. We therefore introduce a measure called JCRASH, which is defined as a stock's conditional probability to realize a left tail event given that *several factors simultaneously* realize a left tail event at the same point in time (instead of conditioning on the occurrence of *at least one factor* crash). Since simultaneous crash events occur very rarely, we restrict our analysis to simultaneous crashes of factor pairs including the market as well as simultaneous crashes of the three Fama and French (1993) factors, the four Carhart (1997) factors and the five Fama and French (2015) factors.[4] We find that there are several factor combinations, for which JCRASH has a significantly positive effect on the cross-section of future returns with MKT *and* SMB as well as MKT *and* RMW as the most pronounced ones. The significant impact of these joint factor crashes remains robust even when we include MCRASH in multivariate regressions.

Our work is related to the theoretical and empirical asset pricing literature on downside risk

[3] These bivariate measures capture the conditional probability that a stock realizes a left tail event given that a specific factor realizes a left tail event at the same point in time.

[4] Due to the very low probabilities of joint factor crashes, we also use a fully parametric copula model instead of a non-parametric model to characterize the dependence structure.

aversion going back to the ideas of Roy (1952) and Markowitz (1959). More specifically, our analysis relates to the literature that examines cross-sectional risk premia for systematic downside and crash risk.[5] Ang et al. (2006a) show that downside beta carries a positive premium in the cross-section of average stock returns.[6] Bollerslev et al. (2021) recently extend the idea behind downside beta and decompose the standard beta into four semibetas, finding a positive premium for high levels of linear dependence in states with negative market and negative stock returns. Focusing on extreme events, Kelly and Jiang (2014) document that stocks with higher loadings on a time-varying tail risk factor have higher future returns. Chabi-Yo et al. (2018) show that a stock's bivariate crash risk as measured by its lower tail dependence with the market carries a positive premium, which is confirmed by Weigert (2016) in a worldwide sample of 40 international countries.[7] In contrast, van Oordt and Zhou (2016) document that a stock's tail beta with the market predicts future performance during market crashes, but does not carry a positive premium. Exploiting forward-looking information from S&P 500 index options, Lu and Murray (2019) show that exposure to changes in the ex-ante probability of market crashes explains average future stock returns. Evidence of crash risk for other asset pricing factors is scarce and is mostly concerned with momentum crashes. Barroso and Santa-Clara (2015) and Daniel and Moskowitz (2016) document that the momentum factor experiences infrequent and persistent strings of negative returns and show that volatility-adjusted momentum strategies have higher Sharpe ratios. Finally, Ruenzi and Weigert (2018) provide a risk-based explanation of the momentum anomaly on equity markets and show that the momentum factor is correlated to market crash risk. We contribute to this strand of literature by analyzing systematic crash risk in a multifactor setting. Our results indicate

[5]Recent studies by Jang and Kang (2019) and Atilgan et al. (2020) document a negative return impact of a stock's *univariate* (non-systematic) crash risk as measured by its probability of price crashes or its value-at-risk.

[6]Lettau et al. (2014) document that the downside risk CAPM also helps to explain risk premia across asset classes. Levi and Welch (2020) raise doubts on the predictive power of downside betas.

[7]The existence of a premium for LTD is also confirmed for other asset classes. Agarwal et al. (2017) find that hedge funds which load on tail risk earn high future returns. Meine et al. (2016) show that bivariate crash risk is compensated in the cross-section of credit default swaps of banks.

that extending the measurement of systematic downside risk to multiple factors improves our understanding of the risk return trade-off in the cross-section of stock returns.

Moreover, we contribute to the literature on the application of non-linear dependence measures in finance. Longin and Solnik (2001) and Poon et al. (2004) apply extreme value theory to study extreme dependencies between selected international equity markets. Patton (2004), Christoffersen et al. (2012) and Christoffersen and Langlois (2013), among others, develop dynamic copula models to describe non-linearities in the conditional dependence structure of asset and factor returns. Our paper is the first to study the asset pricing implications of non-linear dependencies in *multifactor* models. Our results imply that incorporating such dependence features can explain risk premia in the cross-section of stock returns without further extending the range of factors.

We proceed as follows. Section 2 presents a theoretical model for the pricing of multivariate crash risk. Section 3 introduces our data sample and describes the estimation of MCRASH. In Section 4, we document our empirical results on the relationship between multivariate crash risk and average future stock returns. Section 5 investigates alternative notions of multivariate crash risk. Section 6 concludes.

2 Theory

In this section, we introduce our main measure for exposure to multivariate crash risk. We also study the theoretical relationship between multivariate crash risk and expected stock returns using a new expansion of the stochastic discount factor.

2.1 Crash Sensitivity in Multifactor Models

To account for cross-sectional differences in the dispersion of stock and factor returns, we apply a quantile-based definition of crash events. In particular, we fix a small probability level p and consider return realizations at or below the p-quantile as tail events. Formally, we define the p-tail of the random return Y as

$$T_p[Y] := \{Y \leq Q_p[Y]\} \tag{1}$$

with $Q_p[Y] := \sup\{y \in \mathbb{R};\ \mathbb{P}[Y \leq y] \leq p\}$ denoting the upper p-quantile of Y. Note that this understanding of crash events is consistent with quantile-based tail risk measures such as Value-at-Risk or Expected Shortfall, which are heavily used in the risk management and regulation of financial institutions.[8]

In the following, we investigate systematic crash risk in a model with $N \geq 1$ priced factors and denote the returns of these factors over the period $[t, t+1]$ by $\boldsymbol{X} = (X_1, \ldots, X_N)'$. In this setting, the standard univariate definition of tail events given in equation (1) can be generalized as follows: We define a multivariate systematic tail event denoted by $T_p[\boldsymbol{X}]$ as a realization of $\boldsymbol{X}$, where *at least one* of the factors is at or below its p-quantile.[9] Accordingly, $T_p[\boldsymbol{X}]$ can be written as the union of individual factor crashes, i.e.,

$$T_p[\boldsymbol{X}] := \bigcup_{j=1}^{N} T_p[X_j] = \bigcup_{j=1}^{N} \{X_j \leq Q_p[X_j]\}\,. \tag{2}$$

This definition of systematic crash events in multivariate models is rather general. It includes more specific crash types such as the individual factor crashes themselves and simultaneous crashes of several factors, which we will consider later in the paper.[10]

Building on the definition in equation (2), we introduce an asset's crash sensitivity in multifactor models as a straightforward generalization of the well-known bivariate lower tail dependence

[8]Using the upper p-quantile as cut-off point for returns is consistent with using a standard VaR-definition based on the lower $(1-p)$-quantile for the corresponding loss $L = -X$.

[9]Note that this definition is related to the stable tail dependence function of $-\boldsymbol{X}$, which is frequently used in multivariate extreme value theory. See e.g. Beirlant et al. (2004, p. 283), Drees and Huang (1998) or Kiriliouk et al. (2018).

[10]Simultaneous crashes of several factors can be formalized as the intersection of the corresponding individual crash events. See Section 5.2.

coefficients.[11] Let R_i denote the discrete return of an asset over the period $[t, t+1]$. We define the *multivariate crash risk* (MCRASH) of R_i for the factors $\boldsymbol{X}$ at the probability level p by

$$\text{MCRASH}_i^{\boldsymbol{X}} := \mathbb{P}[T_p[R_i] \,|\, T_p[\boldsymbol{X}]] = \mathbb{P}\left[R_i \leq Q_p[R_i] \,\middle|\, \bigcup_{j=1}^{N} \{X_j \leq Q_p[X_j]\}\right], \tag{3}$$

where $\mathbb{P}[A \,|\, B]$ refers to the conditional probability of event A given B. MCRASH thus corresponds to the conditional probability that R_i does not exceed its p-quantile given that at least one of the factors is also at or below its p-quantile. Accordingly, MCRASH measures the probability of asset i to be adversely affected if a crash event occurs for one (or more) of the systematic factors. By construction, MCRASH ranges between 0 and 1. It is high if asset i tends to realize an extreme negative return when the systematic factors realize adverse scenarios and it is low for assets which rarely realize tail events when crashes in the systematic factors occur.

To simplify our theoretical analysis of MCRASH, we add the following technical assumption:

(A1) The univariate distributions of R_i and $X_1, \ldots, X_n$ are continuous with positive densities.

Under this assumption, MCRASH is not influenced by characteristics of the marginal distributions. With F_Y denoting the cumulative distribution function of Y, it holds that $T_p[Y] = T_p[F_Y(Y)]$, where $F_Y(Y)$ is known as the probability integral transform of Y.[12] Therefore, we can rewrite the definition in equation (3) as

$$\text{MCRASH}_i^{\boldsymbol{X}} = \mathbb{P}\left[T_p[F_{R_i}(R_i)] \,\middle|\, \bigcup_{j=1}^{N} T_p[F_{X_j}(X_j)]\right]. \tag{4}$$

This shows that the MCRASH is determined by the copula function of the random vector

[11] See e.g. Poon et al. (2004), Christoffersen et al. (2012) and Chabi-Yo et al. (2018) for applications of bivariate LTD measures in finance.

[12] Since $Q_p[F_Y(Y)] = F_Y(Q_p[Y])$ for continuous and non-decreasing functions F_Y (Dhaene et al., 2002, Theorem 1), $Y \leq Q_p[Y]$ if and only if $F_Y(Y) \leq Q_p[F_Y(Y)]$.

$(R_i, X_1, \ldots, X_N)$, which formally captures the dependence structure of the relevant random variables.[13] This property has two important implications for our analysis: First, by construction, MCRASH is distinct from univariate risk measures such as volatility, skewness, Value-at-Risk or Expected Shortfall. Second, we can resort to standard copula methods to obtain estimates of MCRASH in our empirical analysis.

2.2 Multivariate Crash Risk and Expected Returns

Our theoretical analysis of the risk premium for asset i with the discrete return R_i over the period $[t, t+1]$ relies on a nonnegative stochastic discount factor (SDF) M, which satisfies

$$\mathbb{E}[M\,(1 + R_i)] = 1. \tag{5}$$

The existence of M is guaranteed by no arbitrage (Harrison and Kreps, 1979; Hansen and Richard, 1987). We furthermore assume that there is a risk-free asset, whose discrete return over the period $[t, t+1]$ is given by R_f.

If we can replace M by its projection $M^{\boldsymbol{X}} = \mathbb{E}[M \mid \boldsymbol{X}]$, where $\boldsymbol{X}$ is a set of factors or state variables, then we say that $\boldsymbol{X}$ explains the cross-sectional variation in expected stock returns.[14] Due to its definition as a conditional expectation, the projected SDF can be written as $M^{\boldsymbol{X}} = m(\boldsymbol{X})$ with a measurable function $m : \mathbb{R}^N \to \mathbb{R}$. The function m is often assumed to be linear. Instead of imposing this restriction, we build on the following more general assumption:

(A2) m is differentiable, decreasing in each argument and convex.

Requiring the function m to be decreasing ensures that the factors are defined such that $T_p[\boldsymbol{X}]$ in

[13] The copula function of a random vector $(Y_1, \ldots, Y_N)$ with a continuous distribution can be defined as the distribution function of $(F_{Y_1}(Y_1), \ldots, F_{Y_N}(Y_N))$ (McNeil et al., 2015, Definition 7.5). For a recent review on copula methods, see e.g. Fan and Patton (2014).

[14] This holds if $R_i = f_i(\boldsymbol{X}) + \varepsilon_i$, where f_i is an arbitrary measurable function and ε_i is a zero-mean residual that is not priced conditional on $\boldsymbol{X}$, i.e. $\mathbb{E}[\varepsilon_i \mid \boldsymbol{X}] = 0$ and $\mathrm{cov}[M, \varepsilon_i \,|\, \boldsymbol{X}] = 0$.

equation (2) captures crash (instead of boom) scenarios. An example for a direct specification of a convex SDF in a single factor setting can be found in Harvey and Siddique (2000). In standard representative agent models, the curvature of m depends on the third derivative of the agent's utility function u. If there is a function $g : \mathbb{R}^N \to \mathbb{R}$ that maps the values of the factor realizations into the optimal level of consumption and m is of the form (see, for example, p. 166 in Cochrane, 2005 for a similar structure of the SDF)

$$m(\boldsymbol{X}) = \frac{\delta}{1+R_f}\, u'(g(\boldsymbol{X})) \qquad \text{with} \qquad \delta := \frac{1}{\mathbb{E}[u'(g(\boldsymbol{X})]}, \tag{6}$$

then the linearity of g and $u''' > 0$ are sufficient conditions for the convexity of m.[15] Note that the monotonicity and the convexity restrictions from Assumption (A2) can be relaxed for regions of the factor space that are not relevant to our additional tail-related approximation arguments. Therefore, the convexity requirement is not necessarily inconsistent with the literature on the SDF or risk aversion puzzle (Jackwerth, 2000).

Using the projected SDF $m(\boldsymbol{X})$ and $\mathbb{E}[m(\boldsymbol{X})] = (1+R_f)^{-1}$, it is easy to derive the well-known pricing result

$$\mathbb{E}[R_i - R_f] = -(1+R_f)\,\mathrm{cov}[m(\boldsymbol{X}), R_i]\,. \tag{7}$$

To obtain a beta representation from (7), we can use a first-order Taylor expansion of m around $\boldsymbol{x}_c = \mathbb{E}[\boldsymbol{X}]$ (Cochrane, 2005, p. 161). This linear approximation is given by

$$m_L(\boldsymbol{X}) = m(\boldsymbol{x}_c) + \nabla m(\boldsymbol{x}_c) \cdot (\boldsymbol{X} - \boldsymbol{x}_c), \tag{8}$$

where $\nabla m(\boldsymbol{x}_c) := \left(\frac{\partial m}{\partial x_1}(\boldsymbol{x}_c), \ldots, \frac{\partial m}{\partial x_N}(\boldsymbol{x}_c)\right)$. A drawback of this standard argument is that the approximation quality can be poor for factor realizations that are far away from $\mathbb{E}[\boldsymbol{X}]$, which

[15] See Pratt (1964); Dittmar (2002); Chabi-Yo (2012) for standard assumptions on the derivatives of u.

specifically applies to realizations from the p-tail $T_p[\boldsymbol{X}]$.

To address this issue, we propose a piecewise linear approximation of $m(\boldsymbol{X})$

$$m_{L,e}(\boldsymbol{X}) = m_L(\boldsymbol{X}) + \mathbb{1}(T_p[\boldsymbol{X}])\, d_{\text{tail}}(\boldsymbol{X}), \tag{9}$$

where $\mathbb{1}(T_p[\boldsymbol{X}])$ is the indicator function for the multivariate crash event defined in equation (2) and $d_{\text{tail}}(\boldsymbol{X})$ is a linear adjustment term that improves the approximation quality in the tail region.[16] A very simple choice for $d_{\text{tail}}(\boldsymbol{X})$ is a constant adjustment that corresponds to the approximation error at a point $\boldsymbol{x}_l \in T_p[\boldsymbol{X}]$, i.e.,

$$d_{\text{tail}} \equiv m(\boldsymbol{x}_l) - m_L(\boldsymbol{x}_l). \tag{10}$$

Analogous to $\boldsymbol{x}_c = \mathbb{E}[\boldsymbol{X}]$, one could, e.g., choose $\boldsymbol{x}_l := \mathbb{E}[\boldsymbol{X} \mid T_p[\boldsymbol{X}]]$.[17] We illustrate the piecewise linear approximation resulting from these choices in Figure 2 for an example SDF with $N = 1$. A comparison of this relatively simple approach with a more flexible tail approximation based on a separate Taylor expansion around $\boldsymbol{x}_l$ is presented in Section II of the Internet Appendix.

Building on the specific form of the tail adjustment given in equation (10) and a simple tail approximation for R_i, we derive the extended linear model

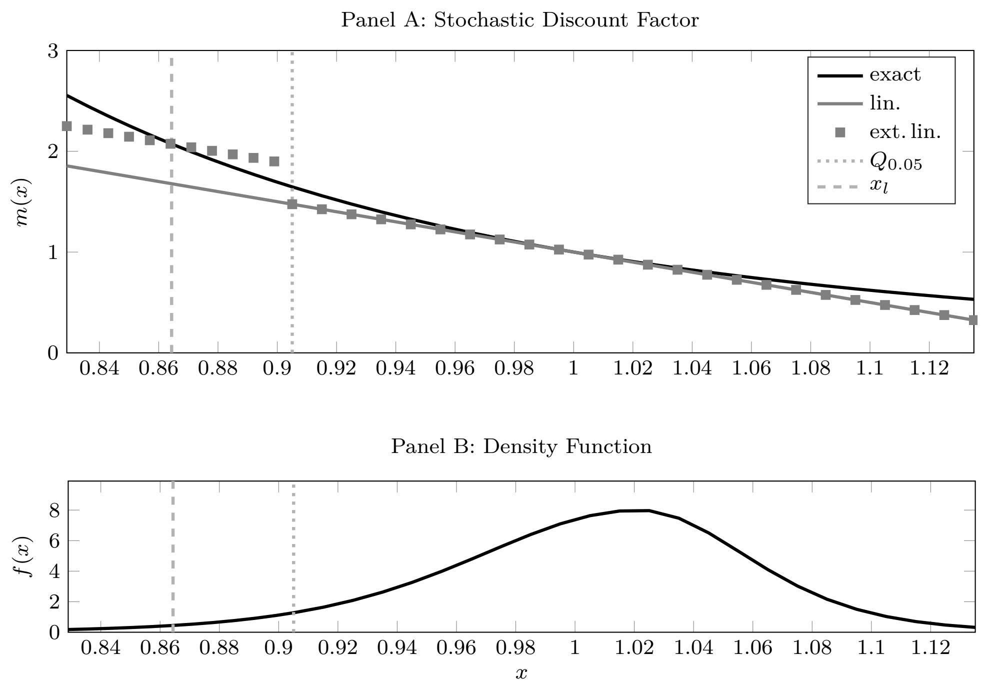
$$\mathbb{E}[R_i - R_f] = \alpha_i + \sum_{j=1}^{N} \beta_i^{(j)}\, \lambda^{(j)} + (\text{MCRASH}_i^{\boldsymbol{X}} - p)\, \lambda_{\text{tail}}^{\boldsymbol{X}} \tag{11}$$

for the expected excess return in Appendix A.1. Equation (11) decomposes the excess return into three components: α_i captures the pricing error arising from our approximation arguments. The sum corresponds to the well-known risk premium in a standard linear model with the factor betas

[16] Diez De Los Rios and Garcia (2011) use a framework consistent with a piecewise linear SDF for assessing and valuing nonlinearities in hedge fund returns.

[17] An optimal choice of $\boldsymbol{x}_l$ is an interesting question, but beyond the scope of the current analysis.

Figure 2: Piecewise Linear SDF Approximation



This figure illustrates our piecewise linear approximation of the SDF for the univariate case $N = 1$ and a standard power utility SDF. Panel A shows the exact (unscaled) discount factor $m(x) = x^{-RRA}$ with a relative risk aversion parameter of $RRA = 5$ (black solid line), its standard linear approximation around $x_c = \mathbb{E}[X]$ (gray solid line) and its piecewise linear approximation according to equation (9) around $x_c = \mathbb{E}[X]$ and $x_l = \mathbb{E}[X \mid X \leq Q_p[X]]$ (gray squares). $p = 0.05$ is used as tail probability threshold in this example. Furthermore, we assume a skewed-t distribution for X with $\mathbb{E}[X] = 1.05$, $\sigma[X] = 0.2$, a skewness parameter of $\lambda = -0.2$ and a degree-of-freedom parameter $\nu = 7$ for this illustration. The density of this distribution is shown in Panel B. The dotted gray line marks the p-quantile Q_p of X and the dashed gray line corresponds to the tail expectation $x_l = \mathbb{E}[X \mid X < Q_p[X]]$.

and the corresponding prices of risk given by

$$\beta_i^{(j)} := \frac{\text{cov}[X_j, R_i]}{\text{var}[X_j]} \qquad \text{and} \qquad \lambda^{(j)} := -(1 + R_f)\,\frac{\partial m}{\partial x_j}(\boldsymbol{x}_c)\,\text{var}[X_j] \tag{12}$$

for $j = 1, \ldots, N$. The last term in equation (11) is an additional crash-related risk premium with MCRASH defined in equation (3) and $\lambda_{\text{tail}}^{\boldsymbol{X}}$ as the associated price of crash risk. $\lambda_{\text{tail}}^{\boldsymbol{X}}$ is non-negative if Assumption (A2) is satisfied as detailed in Appendix A.1.

Based on the extended linear model in equation (11), we arrive at our main hypothesis about the cross-sectional pricing of multivariate crash risk.

Hypothesis: The expected excess return of asset i is increasing in its exposure to multivariate crash risk as measured by $\text{MCRASH}_i^{\boldsymbol{X}}$.

We thus expect higher average returns for assets with high MCRASH coefficients. To be more precise, assets with $\text{MCRASH}_i^{\boldsymbol{X}} > p$ ($\text{MCRASH}_i^{\boldsymbol{X}} < p$) are expected to earn a positive (negative) crash premium compared to the linear factor model.

For $N = 1$ and $p \to 0$, this result provides an alternative explanation of the LTD premium found by Chabi-Yo et al. (2018). Furthermore, the special case of a single factor is also related to the bivariate crash risk measures proposed by Agarwal et al. (2017) and van Oordt and Zhou (2016).

We finally present a numerical example that illustrates the benefits of including MCRASH in linear multifactor models. Our example builds on the form of the SDF given in equation (6) combined with standard CRRA-preferences and a linear mapping function g. To be able to vary non-linear dependence features between R_i and the factor returns $\boldsymbol{X}$, we use a flexible parametric copula model. In particular, we rely on the skewed-t copula introduced by Demarta and McNeil

(2007), which can capture extreme tail dependence as well as dependence asymmetries.[18] Details on the construction of our example are presented in Section III of the Internet Appendix.

Panels A and B in Figure 3 compare the pricing errors arising from a standard first-order approximation according to equation (8) and from our piecewise linear approximation proposed in equations (9) and (10). The graphs are obtained by changing the asymmetry and the tail parameter of the dependence model. We observe that these changes, which do not affect the marginal distributions, cause a non-negligible variation in the pricing errors of the linear model and that the tail-related adjustment term can substantially reduce the magnitude of these errors. Panels C and D in Figure 3 show that there is a positive relationship between the pricing errors of the linear model and MCRASH in line with equation (11).

3 Data and Estimation

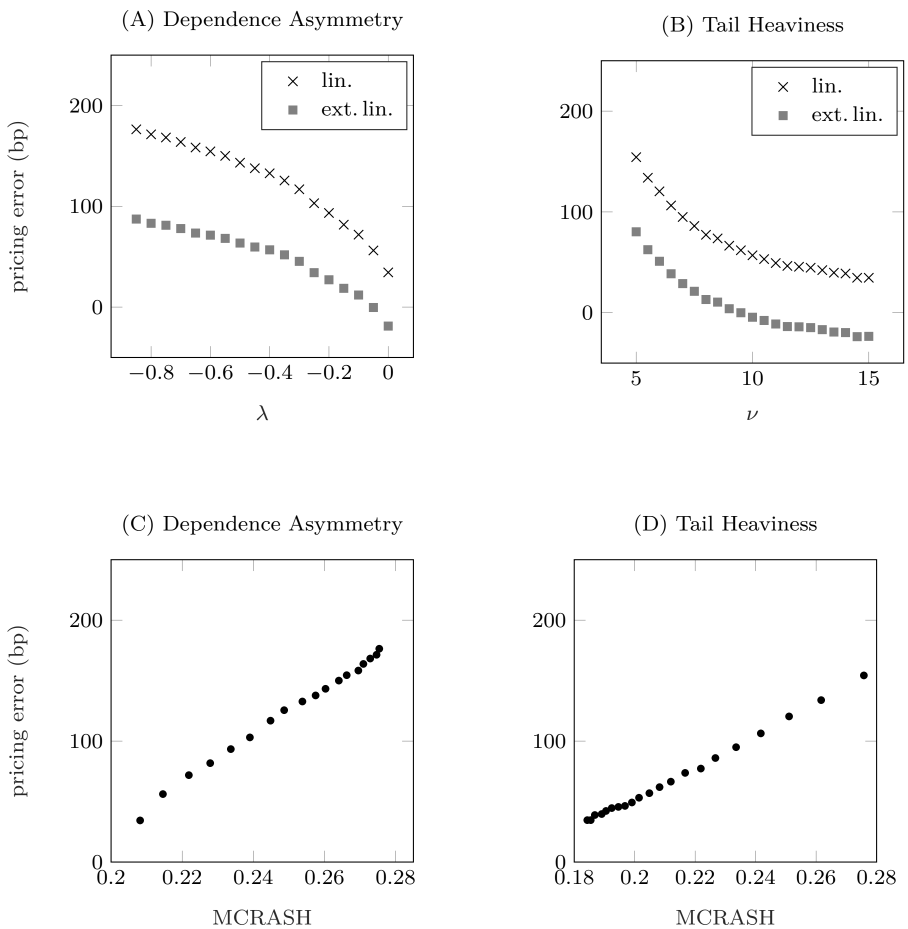
In this section, we first describe the stock sample and the factor data that we use in our empirical analysis. Then, we discuss the estimation of MCRASH and summarize the estimation results obtained for our sample.

3.1 Data

Our sample consists of all common stocks from CRSP (share codes 10 and 11) trading on the NYSE, AMEX, and NASDAQ between January, 1964 through December, 2018. We adjust the raw CRSP holding-period returns obtained from CRSP for delisting events following the procedure proposed by Shumway (1997). To remain in the sample in month t, we require each stock to have at least 200 non-zero return observations over the past 250 trading days and a price of at least USD 2. These filters remove many small and illiquid stocks from our sample. After applying these filters,

[18]Finance applications of this model include Christoffersen et al. (2012) and Christoffersen and Langlois (2013).

Figure 3: Pricing Errors and MCRASH



This figure presents an example that illustrates the relationship between MCRASH and the pricing errors resulting from a linear approximation of the projected SDF. The example builds on the structure of the SDF given in equation (6) with two factors, standard CRRA preferences and a simple linear mapping function g. We use a flexible copula model for the joint distribution of the asset return R_i and the factor returns X_1 and X_2. A detailed description of our assumptions can be found in Section III of the Internet Appendix. Panel A and B compare the pricing errors resulting from a standard linear approximation (crosses) according to equation (8) and the errors from the proposed piecewise linear approximation (squares) according to equations (9) and (10) with $p = 0.05$. Panel A is obtained by changing the asymmetry parameter of the dependence model and, in Panel B, we vary the degrees-of-freedom parameter of the copula that controls the probability of joint tail events. Panel C and Panel D depict the relationship between MCRASH and the pricing errors of the linear model under the same variations of the dependence structure.

our sample consists of 1,477,700 stock-month observations with the number of stocks per month varying between 730 and 4520. In order to account for firms' fundamental information in asset pricing tests in Section 4, we merge our sample with accounting data from Compustat.

In our baseline analysis, we investigate multivariate crash risk in a seven-factor model that combines the five factors proposed by Fama and French (1993, 2015), i.e., the excess market return (MKT), the size factor[19] (SMB), the value factor (HML), the profitability factor (RMW), and the investment factor (CMA), with the UMD momentum factor as in Carhart (1997), and the BAB betting-against-beta factor as in Frazzini and Pedersen (2014).[20] The choice of these factors reflects a common practice in the current asset pricing literature to augment the traditional Fama and French (1993) three-factor model with factors that account for profitability, investment and momentum (see, e.g., Barillas and Shanken, 2018). The inclusion of BAB reflects the considerable attention that low risk anomalies have received recently (see, e.g., Schneider et al., 2020). The stability of our asset pricing results with respect to this choice of factors will be analyzed in our robustness tests in Section 4.4.

3.2 Estimation of MCRASH

We estimate a stock's multivariate crash risk (MCRASH), as defined in equation (3) for each stock i and month t using a rolling window with one year of daily returns. Applying this horizon follows Chabi-Yo et al. (2018) and trades off two concerns: First, a sufficiently large number of observations is needed to obtain reliable estimates for MCRASH. Second, we want to account for the limited stability of risk exposures over long time horizons (see, e.g., Ang and Chen, 2002).

For our baseline analysis, we perform the estimation of MCRASH semiparametrically combining

[19]We use the original size factor from Fama and French (1993) in our analysis. Our empirical results are almost unchanged if we replace SMB with the modification proposed in Fama and French (2015).

[20]Factor data for MKT, SMB, HML, RMW, CMA, and UMD including the risk-free rate are downloaded from the website of Kenneth French; data for BAB are obtained from the AQR homepage.

parametric GARCH models for the marginal return distributions and a non-parametric approach for the dependence modeling. The methodology allows us to account for volatility clustering[21] without imposing a potentially restrictive structure on linear and non-linear dependencies between stock and factor returns.[22] We estimate MCRASH at the 5%-probability level; this means that we estimate the conditional probability that a stock realizes a return observation at or below its 5%-quantile given that at least one of the risk factors is at or below its respective 5%-quantile.[23]

Following the representation of MCRASH in equation (4), we proceed in two steps: First, we estimate GARCH(1,1) models with skewed-t innovations (Hansen, 1994) for the conditional distributions of the daily asset and factor returns over the last 250 trading days. We then apply the resulting marginal cumulative distribution functions to calculate probability integral transforms of the daily returns. In the second step, we evaluate a simple non-parametric estimator of the conditional probability given in equation (4) for the transformed returns calculated in the first step. Accordingly, MCRASH is estimated as the number of days on which asset i and at least one of the factors simultaneously realize a left tail event divided by the overall number of days on which left tail events for the factors occur. A formal description of this estimation procedure is provided in Appendix A.2.

3.3 MCRASH Estimates

We report summary statistics of MCRASH estimates for the end of each month between 1964-12 and 2018-11 in Panel A of Table 1. For the average cross-section, the mean and the median of MCRASH are 0.08 and its standard deviation is 0.03. We also display summary statistics of risk and firm characteristics that are applied as control variables in the asset pricing tests of Section 4.

[21] See, e.g., Poon et al. (2004) for the importance of heteroscedasticity as a source of "tail dependence".

[22] In our robustness checks in Section 4.4, we vary this methodology and also show results obtained from purely non- and fully parametric estimation techniques.

[23] We study the robustness of our results under variations of the tail probability level in Section 4.4.

As risk and firm characteristics, we use betas to the MKT, SMB, HML, RMW, CMA, UMD, and BAB risk factors, size, book-to-market (bm), the past $t-11$ until $t-1$ monthly return (stock-level momentum, mom), the past one-month return (stock-level reversal, rev), illiquidity (illiq), and highest daily return over the past month (max). Definitions of all variables are provided in Table A.1.

Panel B of Table 1 provides correlations between a stock's MCRASH and these different risk and firm characteristics. We observe that correlations are relatively modest: MCRASH is positively correlated to market beta (+0.28), size (+0.21), and the UMD momentum beta (+0.16), while it shows negative correlations to the BAB beta (-0.17), illiquidity (-0.10), and the CMA beta (-0.08). We take particular care of these correlations when investigating the impact of MCRASH on future stock returns in our asset pricing tests in Section 4.

If investors would like to benefit from a risk premium associated with multivariate crash risk, they must be able to forecast the future risk exposure of a stock. We therefore examine the persistence of our MCRASH estimates using multivariate Fama and MacBeth (1973) regressions on the stock level and report the results in Panel C of Table 1. In specifications (1) to (6), we regress future MCRASH in the months $t+1$, $t+3$ and $t+6$ on MCRASH in month t using different sets of risk and firm characteristics as control variables. The coefficient estimate of MCRASH varies from 0.81 to 0.35 and is highly statistically significant (numbers suppressed in the table). The corresponding average adjusted R^2 ranges between 26.86 and 66.89 percent. In specification (6), we test the predictability of MCRASH on the 12 months horizon, so that the corresponding MCRASH estimation windows are non-overlapping. We include the full set of risk measures and firm characteristics. We again find that the MCRASH coefficient is positive and statistically significant at the 1% level with a Newey and West (1987) t-statistic of 9.60 adjusted for 6 lags and an average adjusted R^2 of 13.19 percent.

Table 1: Summary Statistics, Correlations and Persistence

Panel A: Summary Statistics

	Mean	SD	Skew	Kurt	Min	q5	q25	Med	q75	q95	max
MCRASH	0.08	0.03	0.08	2.81	0.00	0.04	0.06	0.08	0.10	0.13	0.17
β^{MKT}	1.04	0.56	0.45	3.86	-0.99	0.23	0.65	0.98	1.37	2.04	3.42
β^{SMB}	0.42	0.78	0.18	4.31	-2.61	-0.76	-0.10	0.40	0.92	1.73	3.91
β^{HML}	-0.75	0.96	-0.53	5.44	-5.42	-2.50	-1.26	-0.63	-0.13	0.56	3.08
β^{RMW}	-0.38	1.11	-0.42	5.77	-6.29	-2.26	-0.98	-0.32	0.29	1.27	4.30
β^{CMA}	-0.89	1.16	-0.62	6.00	-6.63	-3.01	-1.47	-0.73	-0.16	0.68	3.99
β^{UMD}	0.24	0.71	-0.02	5.44	-2.91	-0.87	-0.18	0.21	0.65	1.40	3.31
β^{BAB}	-1.04	0.98	-0.76	5.57	-5.93	-2.87	-1.54	-0.89	-0.37	0.25	2.85
size	5.86	1.67	0.27	2.9	1.37	3.27	4.65	5.79	6.99	8.73	11.72
bm	0.75	0.94	9.78	253.2	0.02	0.14	0.35	0.60	0.93	1.71	24.62
mom	20.44	56.01	4.00	54.8	-79.92	-38.65	-9.03	11.31	36.48	106.29	862.32
rev	1.35	12.16	2.03	38.73	-51.41	-15.29	-5.06	0.62	6.67	19.96	147.17
illiq	0.37	1.52	11.86	263.28	0.00	0.00	0.02	0.06	0.22	1.57	41.81
max	0.06	0.05	6.15	122.86	0.00	0.02	0.03	0.05	0.07	0.13	1.04

Panel B: Correlations

	MCRASH	β^{MKT}	β^{SMB}	β^{HML}	β^{CMA}	β^{RMW}	β^{UMD}	β^{BAB}	size	bm	mom	rev	illiq
MCRASH													
β^{MKT}	0.28												
β^{SMB}	0.06	0.12											
β^{HML}	-0.06	-0.48	-0.01										
β^{RMW}	0.02	-0.13	-0.40	-0.14									
β^{CMA}	-0.08	-0.52	-0.07	0.66	-0.13								
β^{UMD}	0.16	0.15	0.12	-0.16	0.06	-0.11							
β^{BAB}	-0.17	-0.80	-0.08	0.42	0.11	0.42	0.01						
size	0.21	0.11	-0.37	0.00	0.11	0.01	-0.01	-0.10					
bm	-0.05	-0.14	-0.02	0.23	-0.06	0.18	-0.06	0.12	-0.13				
mom	0.07	0.05	0.05	-0.03	0.01	-0.02	0.26	0.00	0.03	0.03			
rev	0.00	-0.02	0.00	0.02	0.01	0.02	0.01	0.02	0.04	0.02	0.01		
illiq	-0.10	-0.14	0.08	0.05	-0.01	0.05	-0.03	0.13	-0.39	0.11	-0.07	-0.02	
max	-0.04	0.23	0.18	-0.17	-0.10	-0.17	0.04	-0.20	-0.31	-0.02	-0.01	0.37	0.16

Table 1: Continued

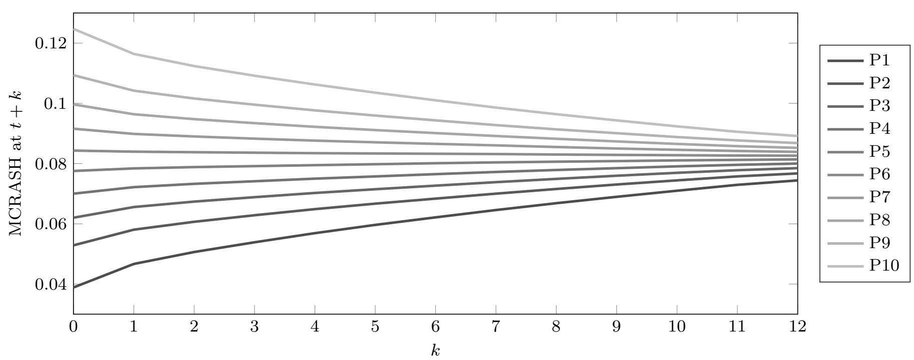
Panel C: Persistence

	future MCRASH						
	$t+1$	$t+1$	$t+1$	$t+1$	$t+3$	$t+6$	$t+12$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
MCRASH(t)	0.81	0.79	0.79	0.77	0.57	0.35	0.05
							(9.60)
betas	no	yes	no	yes	yes	yes	yes
characteristics	no	no	yes	yes	yes	yes	yes
R^2_{adj} [%]	65.86	66.78	66.23	66.89	44.63	26.86	13.19
T	647	647	647	647	645	642	636

Panel A of this table presents summary statistics for MCRASH and for the main control variables used in our asset pricing tests. MCRASH is a firm's multivariate crash sensitivity for a seven-factor model with MKT, SMB, HML, CMA, RMW, UMD and BAB. MCRASH is estimated semiparametrically using a rolling estimation window with 250 days. We combine GARCH skewed-t for the marginal distributions and a non-parametric modeling of the dependence structure. We estimate MCRASH at the probability level $p = 0.05$. As control variables, we include 250-day betas for the same factors and the following stock characteristics: size, the book-to-market ratio (bm), stock-level momentum (mom) and reversal (rev), Amihud (2002) illiquidity (illiq) and max corresponding to the maximum return in month t (max). See Table A.1 for the definitions of these variables. We calculate the mean (Mean), standard deviation (SD), skewness (Skew), excess kurtosis (Kurt), minimum (Min), 5%-quantile (q5), 25%-quantile (q25), median (Med), 75%-quantile (q75), 95%-quantile (q95) and maximum (max). We first calculate these statistics for each cross-section and then take the average over the time domain. Panel B presents the correlation matrix of the variables introduced in Panel A. We report time-series averages of the correlation estimates at the end of each month. Panel C presents results on the persistence of MCRASH with Fama and MacBeth (1973) regressions on the stock level. In specification (1), we regress future MCRASH in month $t+1$ on MCRASH in month t. Specifications (2) and (3) add betas and characteristics as explanatory variables, respectively. Specification (4) simultaneously includes all variables. In columns (5) - (7), we repeat specification (4) with longer predictive horizons. We report a Newey and West (1987) t-statistic with 6 lags for the coefficient estimate in specification (7), which does not use overlapping data for the estimation of MCRASH. Our sample period starts in 1964-12 and ends in 2018-11.

We also analyze the persistence of MCRASH on the portfolio level. For this purpose, we sort all stocks in our sample into ten portfolios according to their level of MCRASH in month t, i.e., stocks with low (high) MCRASH are sorted into portfolio 1 (10). We then track the equal-weighted average of MCRASH for each portfolio over the following months $t+1$ to $t+12$. Figure 4 presents the results.

Figure 4: Persistence of MCRASH



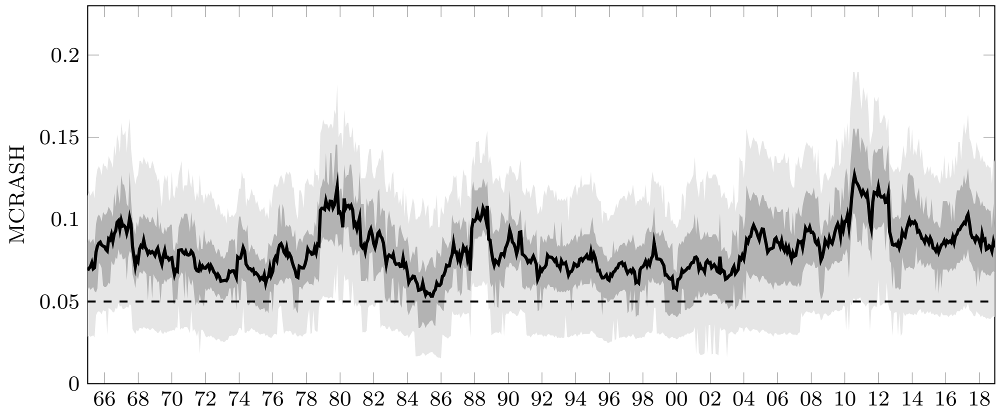
This figure illustrates the persistence of MCRASH estimated for the seven-factor model with $p = 0.05$. We sort all firms in our sample into deciles based on their levels of MCRASH in month t. Then, the equal-weighted average of MCRASH is again computed in the following twelve months $t+1, \ldots, t+12$. We report the time-series average of the corresponding values for each portfolio over our sample period from 1964-12 until 2018-11.

We observe that stocks in decile portfolio 10 consistently show higher MCRASH in the following months than stocks in decile portfolio 1. Consequently, based on the results from Panel C of Table 1 and Figure 4, we document that MCRASH is relatively persistent over short- and medium horizons that will be relevant for our asset pricing tests.

Finally, we analyze the behavior of aggregate MCRASH over time. Aggregate MCRASH in month t is computed as the equal-weighted cross-sectional average over the MCRASH coefficients for all stocks i in this month. Figure 5 plots the time series of aggregate MCRASH over our sample period.

Visual inspection shows that there is no particular trend and that aggregate MCRASH is stationary over time. Furthermore, Figure 5 reveals that the average level of MCRASH is always above

Figure 5: MCRASH over Time



This figure plots the evolution of the multivariate crash risk measure MCRASH estimated with $p = 0.05$ for the seven-factor model over time. The bold black line corresponds to aggregate MCRASH, defined as the equal-weighted average of the estimates at time t over all stocks in our sample. The light (dark) gray area corresponds to the range between the 5%- and the 95%-quantile (the 25%- and the 75%-quantile) across firms. The dashed reference line at 0.05 corresponds to the case that the tail events for R_i and the factors $\boldsymbol{X}$ are independent. Our sample period starts in 1964-12 and ends in 2018-11.

0.05, which is the benchmark value obtained if the crash events for a stock and the factors are independent. The highest spikes in aggregate MCRASH occur between 1979-1980 and 2010-2012, i.e., time periods with high market volatility (possibly triggered by the early 1980s US recession and the worldwide financial crisis from 2007-2009). However, it is important to note that these spikes do not occur during the worst stock market downturns in the USA (i.e., the 1987 Black Monday crash and the 2008 crash related to the Lehman Brothers bankruptcy) indicating that MCRASH is not driven by market crashes alone, but depends to a large extent also on tail events from non-market risk factors.

4 Asset Pricing Implications

The main part of our empirical analysis examines the relationship between multivariate crash risk and average future stock returns for individual stocks in the CRSP sample introduced in the previous section. In particular, we employ MCRASH estimates calculated at the end of each month between December 1964 and November 2018 and returns from January 1965 to December 2018 in our asset pricing tests. To account for the impact of autocorrelation and heteroscedasticity, we determine statistical significance in portfolio sorts and multivariate regressions using Newey and West (1987) standard errors with six lags.

4.1 Univariate Portfolio Sorts

To assess the predictive power of differences in stocks' MCRASH on the cross-section of future stock returns, we first look at simple equal-weighted univariate portfolio sorts. At the end of each month t, we form decile portfolios by sorting stocks based on their multivariate crash risk coefficients, where decile 1 contains stocks with the lowest MCRASH and decile 10 contains stocks with the highest MCRASH. Panel A of Table 2 reports average excess returns (i.e., returns minus the risk-free rate) and alphas of these portfolios in month $t+1$. We also display the differences in (risk-adjusted) returns between portfolio 10 and portfolio 1, i.e., the MCRASH (10)-(1) (risk-adjusted) return spread in the last column of the table.

We observe, moving from decile 1 to decile 10, that average excess returns in month $t + 1$ monotonically increase from 0.38% to 0.77%. This indicates that the MCRASH (10)-(1) return spread amounts to 0.39% per month, which is statistically significant at the 1% level with a t-statistic of 3.69. Hence, stocks in the highest MCRASH decile portfolio earn about 4.68% higher annualized returns than stocks in the lowest MCRASH decile portfolio. In addition, we report risk-adjusted portfolio returns in Panel A of Table 2. Following our theoretical results, our main adjustment is based on the seven factors used in the calculation of MCRASH, i.e., the five factors of Fama and French (2015) (MKT, SMB, HML, RMW, and CMA) as well as the momentum UMD factor of Carhart (1997), and the betting-against-beta BAB factor of Frazzini and Pedersen (2014).

Table 2: Univariate Portfolio Sorts on MCRASH

Panel A: 1-Month Holding Period

	1	2	3	4	5	6	7	8	9	10	10-1
exret	0.38	0.48	0.53	0.59	0.65	0.68	0.68	0.69	0.73	0.77	0.39
	(1.59)	(1.98)	(2.17)	(2.42)	(2.71)	(2.80)	(2.80)	(2.81)	(2.90)	(3.01)	(3.69)
α 7F	-0.28	-0.17	-0.13	-0.07	-0.02	0.01	0.02	0.05	0.11	0.16	0.44
	(-4.73)	(-3.28)	(-2.56)	(-1.65)	(-0.33)	(0.12)	(0.42)	(0.99)	(1.71)	(2.06)	(4.79)
α 5F	-0.33	-0.25	-0.20	-0.14	-0.08	-0.06	-0.04	-0.01	0.04	0.10	0.43
	(-4.55)	(-3.88)	(-3.43)	(-2.73)	(-1.51)	(-1.09)	(-0.75)	(-0.15)	(0.65)	(1.31)	(4.40)
α 7F+LIQ	-0.27	-0.16	-0.12	-0.07	-0.02	0.00	0.02	0.06	0.11	0.15	0.41
	(-4.39)	(-2.88)	(-2.23)	(-1.50)	(-0.44)	(-0.07)	(0.42)	(1.10)	(1.66)	(1.86)	(4.48)
α 7F+QMJ	-0.20	-0.09	-0.06	-0.02	0.04	0.07	0.09	0.12	0.19	0.26	0.47
	(-3.15)	(-1.78)	(-1.24)	(-0.45)	(0.86)	(1.34)	(1.62)	(2.07)	(2.68)	(3.06)	(4.34)
α 7F+STR	-0.31	-0.20	-0.16	-0.10	-0.04	-0.02	0.00	0.03	0.08	0.12	0.43
	(-5.44)	(-3.91)	(-3.17)	(-2.14)	(-0.83)	(-0.42)	(-0.03)	(0.64)	(1.39)	(1.56)	(4.53)
α 7F+LTR	-0.28	-0.17	-0.13	-0.07	-0.01	0.01	0.02	0.06	0.11	0.16	0.44
	(-4.73)	(-3.29)	(-2.57)	(-1.64)	(-0.31)	(0.16)	(0.49)	(1.11)	(1.82)	(2.21)	(5.05)
α M4	-0.15	-0.09	-0.07	-0.03	0.03	0.05	0.06	0.10	0.16	0.23	0.39
	(-1.96)	(-1.33)	(-1.14)	(-0.48)	(0.67)	(1.05)	(1.16)	(1.64)	(2.41)	(2.74)	(3.03)
α q5	-0.11	-0.04	-0.01	0.03	0.07	0.08	0.09	0.13	0.18	0.24	0.35
	(-1.38)	(-0.65)	(-0.24)	(0.51)	(1.17)	(1.15)	(1.24)	(1.63)	(2.00)	(2.22)	(2.70)

Panel B: Cumulative Risk-Adjusted Returns

	1	2	3	4	5	6	7	8	9	10	10-1
1 month	-0.28	-0.17	-0.13	-0.07	-0.02	0.01	0.02	0.05	0.11	0.16	0.44
	(-4.73)	(-3.28)	(-2.56)	(-1.65)	(-0.33)	(0.12)	(0.42)	(0.99)	(1.71)	(2.06)	(4.79)
2 months	-0.49	-0.31	-0.25	-0.16	-0.07	-0.04	0.01	0.11	0.19	0.21	0.70
	(-4.28)	(-3.12)	(-2.71)	(-1.98)	(-0.92)	(-0.46)	(0.12)	(1.19)	(1.81)	(1.59)	(4.02)
3 months	-0.66	-0.45	-0.40	-0.27	-0.14	-0.11	-0.04	0.10	0.14	0.15	0.81
	(-3.89)	(-3.04)	(-3.08)	(-2.31)	(-1.20)	(-0.88)	(-0.35)	(0.76)	(0.96)	(0.85)	(3.30)
4 months	-0.87	-0.60	-0.54	-0.45	-0.34	-0.26	-0.14	-0.01	0.01	0.04	0.92
	(-3.97)	(-3.07)	(-3.10)	(-2.80)	(-2.11)	(-1.52)	(-0.82)	(-0.06)	(0.07)	(0.18)	(3.00)
5 months	-1.01	-0.75	-0.74	-0.68	-0.55	-0.40	-0.23	-0.14	-0.19	-0.12	0.89
	(-3.71)	(-3.20)	(-3.50)	(-3.28)	(-2.63)	(-1.93)	(-1.14)	(-0.67)	(-0.79)	(-0.40)	(2.39)
6 months	-1.14	-0.96	-0.99	-0.94	-0.79	-0.60	-0.41	-0.33	-0.38	-0.30	0.84
	(-3.46)	(-3.35)	(-3.79)	(-3.64)	(-3.15)	(-2.42)	(-1.68)	(-1.34)	(-1.35)	(-0.85)	(1.81)

Table 2: Continued

Panel A of this table reports the results of univariate portfolio sorts based on our multivariate crash risk measure MCRASH for a seven-factor model with MKT, SMB, HML, CMA, RMW, UMD and BAB. At the end of each month t, we rank all stocks in our sample into deciles (1-10) based on their estimated MCRASH and form ten equal-weighted portfolios that we hold over the following month $t + 1$. We report average monthly future excess returns (exret) over the T-Bill and the return of the (10)-(1) spread portfolio that is long in high MCRASH stocks and short in low MCRASH stocks. In addition, we report risk-adjusted returns (alphas) based on the following factor models: the seven factors used for the MCRASH calculation (α 7F), the standard five-factor model (α 5F) and four extended versions of the seven factor model, in which we add the Pástor and Stambaugh (2003) traded liquidity factor (α 7F+LIQ), the Asness et al. (2019) quality minus junk factor (α 7F+QMJ), the short term reversal factor (α 7F+STR) and the long term reversal factor (α 7F+LTR). Moreover, we report alphas based on the Stambaugh and Yuan (2017) four-factor model (α M4) and on the Hou et al. (2021) q-factor model (α q5). Panel B reports the same results for risk-adjusted cumulative returns over 1-6 months. For the risk adjustment, we use the seven-factor model (which is also applied for the MCRASH calculation). Our return sample covers the period from 1965-01 until 2018-12. We report t-statistics computed using Newey and West (1987) standard errors with 6 monthly lags in parentheses.

In addition, we consider risk-adjustments with the Fama and French (2015) five-factor model, the seven-factor model extended by the Pástor and Stambaugh (2003) traded liquidity LIQ factor, the Asness et al. (2019) quality-minus-junk QMJ factor, the Fama and French short-term STR and long-term LTR reversal factors, as well as the Stambaugh and Yuan (2017) four-factor model (M4) and the Hou et al. (2021) q-factor model (q5).[24]

We find that, in all specifications, the risk-adjusted return of the MCRASH-spread portfolio remains positively significant at the 1% level.[25] For our baseline risk-adjustment using the seven-factor model, the alpha of the spread portfolio amounts to annualized 5.28% and is statistically significant at the 1% level with a t-statistic of 4.79. Hence, adjusting for linear risk exposure leads to an even higher premium for MCRASH compared to evaluating excess returns.

Is the significant return difference due to the outperformance of high MCRASH stocks, or due to the underperformance of low MCRASH stocks, or both? Panel A of Table 2 shows that the seven-factor alphas of decile 1 and decile 10 are -0.28% and 0.16% with t-statistics of -4.73 and

[24]Definitions of all risk factors are presented in Table A.1. Summary statistics for the risk factor time series are provided in Table IA.1 in the Internet Appendix.

[25]Detailed factor loadings of the MCRASH-spread portfolio on the respective risk factors are shown in Table IA.2 in the Internet Appendix.

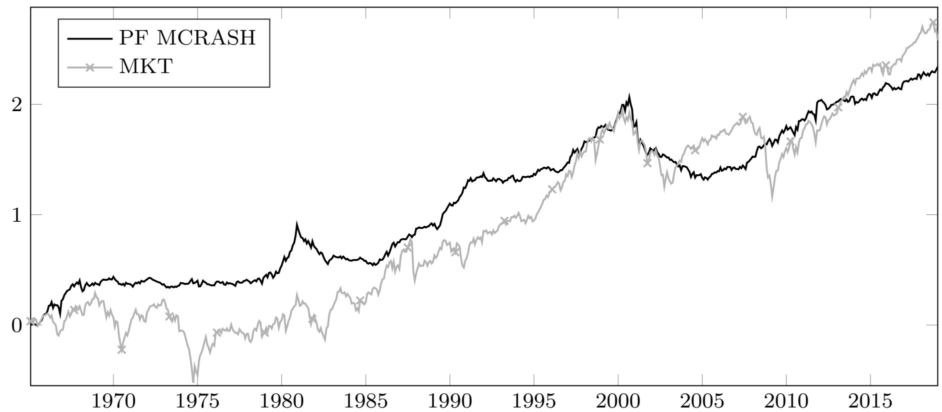
2.06. Hence, we conclude that both the short and the long leg contribute to the outperformance of the MCRASH (10)-(1) spread portfolio. This result is consistent with our theory, which predicts that stocks with MCRASH $< p$ should have negative alphas and stocks with MCRASH $> p$ should have positive alphas.

We illustrate the raw cumulative performance of a long-short investment strategy that buys stocks in decile portfolio 10 and sells stocks in decile portfolio 1 from 1965 to 2018 in Figure 6. As a comparison, we also draw the cumulative performance of the MKT factor. Visual inspection of Figure 6 indicates that the MCRASH long-short trading strategy earns a cumulative return similar to the excess return of the market with a lower overall standard deviation. Interestingly, Figure 6 shows that substantial market crashes are not always directly transmitted to crashes in the MCRASH long-short trading strategy. For example, during the two worst stock market crashes in our sample (i.e., October 1987 and October 2008 with monthly excess market returns of -23.24% and -17.23%), the MCRASH long-short trading strategy earned *positive* returns of +1.82% and +0.47%. Although surprising at first sight, this is feasible since the latter trading strategy focuses on crashes of the market *and* non-market risk factors (with the market being one risk factor out of seven).

To assess the predictive power of MCRASH on future stock returns in the medium-term, we repeat univariate portfolio sorts on MCRASH with longer holding periods. In particular, we calculate cumulative returns for holding periods between 1 and 6 months and present the corresponding seven-factor alphas in Panel B of Table 2. We find that the risk-adjusted MCRASH (10)-(1) return spread becomes slightly weaker when we evaluate longer return horizons; nevertheless, it remains economically and statistically significant (at least at the 10% level) up to six months into the future.

In summary, results from asset pricing tests based on univariate portfolio sorts document a strong positive relationship between MCRASH and future (risk-adjusted) average stock returns.

Figure 6: Performance of the MCRASH Trading Strategy



This figures displays the performance of a long-short investment strategy that buys stocks with high levels of MCRASH and sells stocks with low levels of MCRASH. It shows the cumulative log-return of the equal weighted (10)-(1) MCRASH spread portfolio whose construction is detailed in Section 4.1. We include the performance of the excess market return for comparison. Our sample period is from 1965-1 until 2018-12.

We now turn to a multivariate approach to check the stability of this relationship when we control for conventional linear risk exposures, firm characteristics, and alternative downside risk measures.

4.2 Multivariate Analysis

We run Fama and MacBeth (1973) regressions on the individual firm level. More specifically, we regress a stock's excess return R^e_{it+1} in month $t+1$ on its multivariate crash sensitivity $\text{MCRASH}_{i|t}$ estimated at the end of month t, i.e., we implement

$$R^e_{it+1} = \lambda^0_{t+1} + \lambda^{\text{CRASH}}_{t+1} \cdot \text{MCRASH}_{i|t} + \sum_{j=1}^{K} \lambda^j_{t+1} \, Y^j_{i|t} + \varepsilon_{it+1} \tag{13}$$

with $Y^j_{i|t}$, $j = 1, \ldots, K$, denoting a selection of control variables for stock i measured at the end of month t.[26] To mitigate the impact of outliers, we winsorize all independent variables at the 0.005 probability level in all regressions.

[26]In our baseline results, we use OLS for the cross-sectional regressions. WLS results applying the gross return weighting scheme proposed by Asparouhova et al. (2013) are shown in Table IA.3 of the Internet Appendix.

Motivated by the theory developed in Section 2.2, we first analyze specifications with linear factor betas as controls. Specification (1) in Table 3 summarizes univariate regressions of future excess returns on MCRASH. In accordance with the univariate portfolio sorts documented in Panel A of Table 2, we find a positive coefficient estimate of 4.33 which is statistically significant at the 1% level with a t-statistic of 3.58. In specifications (2) to (8), we incrementally add β^{MKT}, β^{SMB}, β^{HML}, β^{RMW}, β^{CMA}, β^{UMD}, and β^{BAB} as control variables. We find that, in all models, the impact of MCRASH on future returns is statistically significant at the 1% percent level with t-statistics ranging from 5.42 up to 5.90. Hence, in line with the extended linear model presented in equation (11), we confirm that the price impact of MCRASH is not subsumed by linear factor betas.

We continue our investigation by examining whether our results are affected by the inclusion of firm characteristics. In addition to the standard market beta, we include a firm's size (Banz, 1981), book-to-market (Basu, 1983), the cumulative return from month $t-11$ until $t-1$ to account for momentum (Jegadeesh and Titman, 1993), the past one-month return to account for short-term reversal (Jegadeesh, 1990), illiquidity (Amihud, 2002), and max, the highest past daily return over the past month (Bali et al., 2011). A more detailed description of these variables is given in Table A.1.

Our empirical results reported in specifications (1) to (8) of Table 4 confirm several findings from the literature: book-to-market, momentum, and illiquidity are positive predictors of future returns, while the price impact of reversal and size is negative. More importantly in our context, we find that the inclusion of different firm characteristics does not subsume the impact of MCRASH on future returns. Our results reveal that the coefficient estimate for MCRASH is between 2.69 and 5.56 with t-statistics ranging from 3.58 and 6.70. In specification (9), we finally add the linear factor betas to our model: Again, our results remain robust and indicate a highly significant impact

Table 3: Fama and MacBeth (1973) Regressions with Betas

	future excess returns							
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
MCRASH	4.33	5.56	5.00	5.41	5.04	4.80	4.45	4.37
	(3.58)	(5.77)	(5.42)	(5.76)	(5.90)	(5.82)	(5.90)	(5.89)
β^{MKT}		-0.25	-0.22	0.04	0.16	0.20	-0.01	0.05
		(-1.54)	(-1.19)	(0.18)	(0.71)	(0.87)	(-0.06)	(0.21)
β^{SMB}			-0.03	-0.04	0.11	0.12	0.15	0.09
			(-0.30)	(-0.32)	(0.87)	(0.97)	(1.21)	(0.73)
β^{HML}				0.20	0.24	0.09	0.00	0.02
				(1.84)	(1.77)	(0.75)	(0.02)	(0.15)
β^{RMW}					0.18	0.21	0.24	0.26
					(2.34)	(2.71)	(3.52)	(3.55)
β^{CMA}						0.14	0.11	0.09
						(1.47)	(1.36)	(1.13)
β^{UMD}							-0.09	0.01
							(-0.36)	(0.05)
β^{BAB}								-0.01
								(-0.09)
Intercept	0.22	0.38	0.42	0.33	0.34	0.34	0.36	0.36
	(0.88)	(1.81)	(2.04)	(1.69)	(1.73)	(1.79)	(1.91)	(2.00)
R^2_{adj} [%]	0.41	3.00	4.12	4.77	5.17	5.45	5.91	6.13
$\bar{n}$	2280	2280	2280	2280	2280	2280	2280	2280

This table presents the results of multivariate Fama and MacBeth (1973) regressions of future excess returns over the risk-free rate on MCRASH controlling for linear risk exposures. MCRASH is calculated for a seven-factor model with MKT, SMB, HML, CMA, RMW, UMD and BAB. We control for the linear risk exposures to the seven factors as measured by their betas. The betas are estimated with a rolling window of 250 daily returns. All independent variables are winsorized at the 0.005 probability level. Our return sample covers the period 1965-01 until 2018-12. We report t-statistics calculated using Newey and West (1987) standard errors with 6 monthly lags in parentheses.

Table 4: Fama and MacBeth (1973) Regressions with Firm Characteristics

	future excess returns								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
MCRASH	4.33	5.56	4.80	4.26	3.46	3.37	3.36	2.69	2.78
	(3.58)	(5.77)	(6.70)	(5.80)	(4.85)	(4.76)	(4.68)	(3.96)	(4.82)
β^{MKT}		-0.25	-0.27	-0.19	-0.28	-0.30	-0.32	-0.11	0.21
		(-1.54)	(-1.56)	(-1.13)	(-1.83)	(-1.90)	(-2.10)	(-0.78)	(0.93)
size			-0.01	-0.01	-0.01	0.01	-0.02	-0.09	-0.10
			(-0.15)	(-0.14)	(-0.22)	(0.24)	(-0.55)	(-2.27)	(-2.69)
bm				0.27	0.25	0.28	0.27	0.23	0.20
				(3.40)	(3.30)	(3.49)	(3.24)	(2.80)	(2.76)
mom					0.01	0.01	0.01	0.01	0.01
					(5.72)	(5.23)	(4.87)	(4.80)	(6.02)
rev						-0.03	-0.03	-0.02	-0.03
						(-8.00)	(-7.90)	(-4.96)	(-5.89)
illiq							0.01	0.05	0.03
							(0.13)	(0.93)	(0.60)
max								-9.39	-8.70
								(-10.46)	(-10.68)
betas	no	no	no	no	no	no	no	no	yes
Intercept	0.22	0.38	0.44	0.23	0.22	0.12	0.40	1.16	1.11
	(0.88)	(1.81)	(1.21)	(0.62)	(0.63)	(0.33)	(1.06)	(3.34)	(3.52)
R^2_{adj} [%]	0.41	3.00	4.45	5.15	6.15	6.79	6.99	7.29	9.03
$\bar{n}$	2280	2280	2280	1911	1910	1910	1868	1868	1868

This table presents the results of multivariate Fama and MacBeth (1973) regressions of future excess returns over the risk-free rate on MCRASH controlling for standard firm characteristics. MCRASH is calculated for a seven-factor model with MKT, SMB, HML, CMA, RMW, UMD and BAB. We control for a stock's market beta β^{MKT} as well as size, book-to-market (bm), stock-level momentum (mom), stock-level reversal (rev), the Amihud (2002) illiquidity measure (illiq) and the maximum return in a given month (max). In specification (9), we simultaneously control for stock characteristics and factor betas (applied in Table 3). These characteristics are defined in Table A.1. All independent variables are winsorized at the 0.005 probability level. Our return sample covers the period from 1965-01 until 2018-12. We report t-statistics calculated using Newey and West (1987) standard errors with 6 monthly lags in parentheses.

of MCRASH on future returns when controlling for all risk and firm characteristics used in Tables 3 and 4.

To assess the economic significance of these results, we include information from both univariate portfolio sorts and Fama and MacBeth (1973) regressions. We first compute the difference between average MCRASH for decile portfolios 10 and 1 given by 0.12 - 0.04 = 0.08. Multiplying this spread by the average slope coefficients in the regressions of Table 3 and Table 4 yields estimated annualized premiums between 2.58% and 5.34%.

We also investigate the impact of MCRASH on future risk-adjusted returns when controlling for β^{MKT} and firm characteristics in bivariate portfolio sorts. To do so, we first form quintile portfolios on the respective control variable in month t. Then, within each quintile and month, we sort stocks into five portfolios based on MCRASH. Our results are summarized in Table 5. To save space, we only report risk-adjusted returns for the spread portfolios that are long in MCRASH quintile portfolio 5 (stocks with high MCRASH) and short in MCRASH quintile portfolio 1 (stocks with low MCRASH) for each of the quintiles sorted on beta and firm characteristics together with the risk-adjusted return of the average spread portfolio. For all control variables, the risk-adjusted returns of the average MCRASH spread portfolios are statistically and economically significant with seven-factor alphas ranging from 0.22% per month (when controlling for book-to-market) to 0.33% per month (when controlling for market beta).[27] Similar results are obtained for the corresponding returns and in bivariate sorts that control for non-market betas as shown in Tables IA.4 and IA.5 of the Internet Appendix.

We conclude that the impact of MCRASH on the cross-section of average stock returns is not explained by traditional factor betas and firm characteristics.

[27] An interesting finding based on the bivariate portfolio sorts is that the premium for multivariate crash risk is stronger for stocks with low book-to-market values than for stocks with high book-to-market values. This finding is in line with previous research documenting that value stocks are risky (Zhang, 2005; Petkova and Zhang, 2005; Galsband, 2012) and implies that the premiums for value and multivariate crash risk could potentially overlap.

Table 5: Bivariate Portfolio Sorts – MCRASH and Characteristics

	Control Portfolio					
Control	1	2	3	4	5	avg
β^{MKT}	0.19	0.28	0.41	0.48	0.31	0.33
	(2.62)	(3.87)	(5.15)	(5.78)	(2.65)	(6.61)
size	0.44	0.34	0.29	0.11	0.19	0.27
	(3.83)	(3.04)	(3.27)	(1.35)	(2.37)	(4.00)
bm	0.51	0.30	0.17	0.14	0.01	0.22
	(4.47)	(3.03)	(1.81)	(1.80)	(0.11)	(3.53)
mom	0.49	0.21	0.19	-0.03	0.28	0.23
	(4.49)	(2.23)	(2.29)	(-0.36)	(2.94)	(3.65)
rev	0.43	0.31	0.25	0.23	0.36	0.31
	(3.77)	(3.02)	(3.02)	(2.35)	(3.80)	(4.81)
illiq	0.18	0.14	0.38	0.40	0.36	0.29
	(1.96)	(1.55)	(3.66)	(4.35)	(3.63)	(4.25)
max	0.18	0.08	0.25	0.34	0.65	0.30
	(2.74)	(1.19)	(2.82)	(3.14)	(4.87)	(4.98)

This table summarizes the results of bivariate portfolio sorts on MCRASH, in which we control for market beta and stock characteristics. MCRASH is calculated for a seven-factor model with MKT, SMB, HML, RMW, CMA, UMD and BAB. We consider the same control variables as in Table 4. At the end of each month t, we first sort the stocks in our sample into quintile portfolios according to one of these control variables. Then, we sort the stocks within each of the quintile portfolios obtained from the first step according to their MCRASH coefficients into five sub-portfolios. We report the seven-factor alphas of the (5) - (1) MCRASH spread portfolios for each the five portfolios formed on the control variables. Furthermore, we show the seven-factor alpha for the average (5) - (1) spread portfolio in the last column. The return sample covers the period from 1965-01 until 2018-12. We report t-statistics computed using Newey and West (1987) standard errors with 6 monthly lags in parentheses.

4.3 Alternative Downside Risk Measures

A potential concern is that MCRASH may be correlated with alternative downside and tail risk measures from the literature that have already been shown to explain expected stock returns. To address this concern, we again run multivariate regressions of future excess returns on MCRASH and control for the following downside and tail risk measures: a stock's downside beta β^{down} (Ang et al., 2006a), the tail beta β^{tail} proposed by Kelly and Jiang (2014), idiosyncratic volatility and idiosyncratic skewness (Ang et al., 2006b), coskewness and cokurtosis (Harvey and Siddique, 2000), Value-at-Risk (Atilgan et al., 2020), and the Lu and Murray (2019) bear beta β^{bear}. In order to be consistent with the computation of MCRASH, we estimate the additional measures except β^{tail} and β^{bear} with a rolling estimation window of 250 daily returns. More details on the calculation of the additional risk measures are shown in Table A.1 and summary statistics are provided in Table IA.6 of the Internet Appendix.[28] We report our regression results in Table 6.

Specification (1) repeats specification (8) of Table 4 including all firm characteristics as the baseline. In each of the specifications (2) to (9), we add one of the alternative downside risk measures to the baseline model. In line with the previous literature, we find that a stock's tail beta and cokurtosis have a positive impact, whereas idiosyncratic volatility, value-at-risk, and bear beta show a negative effect. Turning to the impact of MCRASH, we find coefficient estimates ranging from 1.49 to 3.57 with t-statistics between 2.48 and 4.60 indicating a statistically significant effect at least at the 5% percent level.[29]

We also investigate the impact of MCRASH on future (risk-adjusted) returns when controlling for the same set of alternative downside and tail risk measures in bivariate portfolio sorts. We again apply the bivariate sorting methodology described in Section 4.2. Our results are summarized in

[28]Table IA.7 of the Internet Appendix shows that our results are robust if we alter the estimation methodology of the downside and tail risk measures.

[29]Note that the sample period for regression (9) is restricted to 1997-01 until 2015-09 due to the availability of data as in Lu and Murray (2019). We thank Scott Murray for providing the AD Bear excess returns.

Table 6: Fama and MacBeth (1973) Regressions with Downside Risk Measures

	future excess returns								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
MCRASH	2.69	3.20	2.46	1.49	2.73	2.94	1.95	1.57	3.57
	(3.96)	(4.60)	(3.60)	(2.48)	(4.01)	(4.47)	(3.21)	(2.56)	(2.76)
β^{down}		-0.19							
		(-2.55)							
β^{tail}			0.16						
			(1.92)						
idiovol				-0.02					
				(-5.74)					
idioskew					-0.02				
					(-0.64)				
coskew						0.14			
						(0.71)			
cokurt							0.28		
							(3.75)		
VaR								-26.97	
								(-5.37)	
β^{bear}									-0.29
									(-2.65)
Characteristics	yes	yes	yes	yes	yes	yes	yes	yes	yes
$R^2_{\text{adj}}[\%]$	7.29	7.45	7.51	7.82	7.40	7.40	7.52	7.80	5.97
$\bar{n}$	1868	1868	1670	1868	1868	1868	1868	1868	3124
T	648	648	648	648	648	648	648	648	225

This table presents the results of multivariate Fama and MacBeth (1973) regressions of future excess returns over the risk-free rate on MCRASH, controlling for firm characteristics and downside risk measures. MCRASH is calculated for a seven-factor model with MKT, SMB, HML, RMW, CMA, UMD and BAB. The standard controls correspond to specification (8) in Table 4. As additional downside risk measures, we include the Ang et al. (2006a) downside beta (β^{down}), the Kelly and Jiang (2014) tail beta (β^{tail}), idiosyncratic volatility (idiovol), idiosyncratic skewness (idioskew), coskewness (coskew), cokurtosis (cokurt), the 5%-Value-at-Risk (VaR), and the Lu and Murray (2019) bear beta (β^{bear}). See Table A.1 for details on the definition and estimation of these risk measures. The independent variables are winsorized at the 0.005 probability level. Our return sample covers the period 1965-01 until 2018-12. The sample period for specification (9) is restricted to 1997-01 until 2015-09 due to the availability of data as in Lu and Murray (2019). We report t-statistics computed using Newey and West (1987) standard errors with 6 monthly lags in parentheses.

Table 7. We report the seven-factor alphas of the spreads between MCRASH quintile portfolio 5 (stocks with high MCRASH) and MCRASH quintile portfolio 1 (stocks with low MCRASH) for each alternative downside and tail risk quintile together with the risk-adjusted return of the respective average spread portfolio. Our results reveal that, when explicitly controlling for alternative downside and tail risk measures, the average MCRASH (5)-(1) spread portfolio yields alphas ranging from 0.19% per month (when controlling for idiosyncratic volatility or value-at-risk) to 0.32% per month (when controlling for idiosyncratic skewness). These alphas are statistically significant at the 1%-level except for the alpha obtained when controlling for bear beta, which is only significant at the 10%-level.[30]

To summarize, our empirical findings provide strong evidence that investors care about multivariate crash risk of stocks. The MCRASH return premium is not explained by linear risk factor exposure or firm characteristics; it is also different from the impact of alternative downside and tail risk measures. We document that accounting for multivariate crash risk to established state variables helps to determine the cross-section of expected stock returns without further expanding the factor zoo.

4.4 Stability Checks

In this section, we summarize the results from a battery of additional stability checks.

Our asset pricing tests based on univariate portfolio sorts in Section 4.1 are performed on an equal-weighted basis. Thus, our results could be influenced by overweighting the importance of relatively small stocks and we now examine univariate value-weighted portfolio sorts.[31] Panel A of Table 8 reports seven-factor alphas of the decile portfolios and the spread portfolio; results for

[30]Again, note that the sample period for portfolio sorts based on β^{bear} and MCRASH is restricted to 1997-01 until 2015-09. Similar results are obtained for excess returns as shown in Table IA.8 of the Internet Appendix.

[31]Remember that many very small and illiquid stocks are already excluded from our sample through the initial requirement to have at least 200 non-zero return observations over the past 250 trading days.

Table 7: Bivariate Portfolio Sorts – MCRASH and Downside Risk Measures

	Control Portfolio					
Control	1	2	3	4	5	avg
β^{down}	0.18	0.19	0.33	0.22	0.51	0.29
	(2.81)	(2.69)	(4.76)	(2.47)	(4.30)	(5.54)
β^{tail}	0.42	0.25	0.20	0.14	0.36	0.27
	(3.33)	(2.91)	(2.38)	(1.47)	(3.70)	(3.91)
idiovol	0.05	0.03	0.15	0.14	0.55	0.19
	(0.95)	(0.53)	(2.12)	(1.27)	(3.88)	(3.08)
idioskew	0.29	0.22	0.16	0.40	0.52	0.32
	(3.58)	(2.57)	(1.85)	(4.05)	(5.07)	(4.55)
coskew	0.18	0.39	0.19	0.32	0.28	0.27
	(1.80)	(4.25)	(1.95)	(3.44)	(2.54)	(3.88)
cokurt	0.28	0.24	0.12	0.27	0.23	0.23
	(3.60)	(3.12)	(1.70)	(2.77)	(2.65)	(4.34)
VaR	0.02	0.02	0.11	0.26	0.56	0.19
	(0.31)	(0.40)	(1.57)	(2.70)	(3.88)	(3.60)
β^{bear}	0.29	0.20	0.10	0.28	0.17	0.21
	(1.79)	(1.32)	(0.83)	(1.72)	(0.98)	(1.90)

This table summarizes the results of bivariate portfolio sorts on MCRASH, in which we control for alternative downside risk measures. MCRASH is calculated for a seven-factor model with MKT, SMB, HML, RMW, CMA, UMD and BAB. At the end of each month t, we first sort the stocks in our sample into quintile portfolios according to each of the alternative downside risk measures considered in Table 6. Then, we sort the stocks within each of the quintile portfolios obtained from the first step according to their MCRASH coefficients into five sub-portfolios. We report the average seven-factor alphas of the (5) - (1) MCRASH spread portfolios for each the five portfolios formed on the alternative downside risk measures. Furthermore, we show the seven-factor alpha of the average (5) - (1) spread portfolio in the last column. Our return sample covers the period from 1965-01 until 2018-12 for all sorts except β^{bear}, for which we again use 1997-01 until 2015-09. We report t-statistics computed using Newey and West (1987) standard errors with 6 monthly lags in parentheses.

excess returns are very similar and are reported in Table IA.9 of the Internet Appendix. We observe that, when we perform univariate sorts based on all stocks in the cross-section, the value-weighted MCRASH (10)-(1) risk-adjusted return spread is smaller than for equal-weighted sorts and amounts to monthly 0.16%, which is not statistically significant at the 10% level.

To investigate this empirical finding more closely, we perform value-weighted portfolio sorts, but we exclude the largest stocks in the cross-section. Our results reveal that, when excluding the top 1% largest stocks in our sample at the end of each month, the MCRASH (10)-(1) risk-adjusted return spread increases to 0.26% per month (3.12% per annum) and is statistically significant at the 5% level. Continuing in this manner and excluding the largest 5% (10%, 20%) of stocks in the cross-section, leads to a risk-adjusted future return spread between stocks with high and low MCRASH of 0.26% (0.30%, 0.37%) with a t-statistic of 2.79 (3.22, 3.79). We conclude that it is the subset of very large (top 1%) stocks for which the impact of MCRASH is negligible; the significantly positive relationship between MCRASH and average future returns is robust for all other size-categories in the cross-section of stocks.[32]

We next examine the stability of the relationship between MCRASH and future returns when we vary our estimation methodology and the sample data used in our analysis. For this purpose, we again run Fama and MacBeth (1973) regressions using specification (8) of Table 4 that controls for β^{MKT} and all stock characteristics. Panels B and C of Table 8 summarize the results of these additional regressions. We only display the coefficient estimate on MCRASH.

Specifications (2) and (3) of Panel B show that our main results do not hinge upon the exact probability level p used to determine the cut-off value for a left tail event. We also obtain positive and significant results for the relation between MCRASH and future returns when we apply $p = 2.5\%$ or $p = 10\%$ (instead of $p = 5\%$). In specifications (4) and (5), we document that our

[32] This result is related to Hou et al. (2020), who show that the price impact of risk factors is particularly pronounced for the subset of small and medium-sized stocks and weaker for stocks with a very large market capitalization.

Table 8: Robustness

Panel A: Value-Weighted Sorts Risk-Adjusted Returns

	1	2	3	4	5	6	7	8	9	10	10-1
all	-0.11	-0.03	-0.05	-0.09	0.00	0.05	0.03	-0.01	-0.04	0.05	0.16
	(-1.25)	(-0.46)	(-1.12)	(-2.00)	(-0.01)	(1.29)	(0.86)	(-0.13)	(-0.90)	(0.97)	(1.39)
ex 1%	-0.15	-0.11	-0.10	-0.10	-0.04	0.02	0.01	0.01	0.06	0.11	0.26
	(-1.89)	(-1.95)	(-2.17)	(-2.39)	(-0.84)	(0.45)	(0.28)	(0.22)	(1.17)	(1.80)	(2.47)
ex 5%	-0.16	-0.12	-0.09	-0.07	-0.02	0.01	-0.01	0.02	0.07	0.10	0.26
	(-2.14)	(-1.94)	(-1.66)	(-1.55)	(-0.53)	(0.13)	(-0.25)	(0.40)	(1.23)	(1.58)	(2.79)
ex 10%	-0.19	-0.12	-0.05	-0.03	-0.01	0.02	0.01	0.02	0.05	0.11	0.30
	(-2.75)	(-2.06)	(-0.86)	(-0.59)	(-0.11)	(0.32)	(0.15)	(0.30)	(0.92)	(1.52)	(3.22)
ex 20%	-0.24	-0.15	-0.06	-0.03	-0.01	0.01	0.01	0.02	0.06	0.12	0.37
	(-3.48)	(-2.30)	(-1.12)	(-0.55)	(-0.11)	(0.20)	(0.11)	(0.34)	(1.10)	(1.55)	(3.79)

Panel B: Estimation Methods

	(1) base	(2) 10% tail	(3) 2.5% tail	(4) non par.	(5) fully par.	(6) 500d marg	(7) 1000d marg	(8) GJR marg	(9) normal GARCH	(10) DCC
MCRASH	2.69	2.02	1.60	1.86	13.08	2.19	2.21	2.47	2.66	7.51
	(3.96)	(2.17)	(2.92)	(2.58)	(3.07)	(2.97)	(3.08)	(3.76)	(3.85)	(1.89)
Characteristics	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
R^2_{adj} [%]	7.29	7.31	7.28	7.30	7.54	7.30	7.30	7.30	7.30	7.55

Panel C: Data & Filtering

	(1) base	(2) 1965-1991	(3) 1992-2018	(4) no filter	(5) mcap 20% NYSE	(6) mcap 50% NYSE	(7) 49 Industries
MCRASH	2.69	2.25	3.14	2.14	2.07	2.07	2.43
	(3.96)	(2.65)	(2.97)	(3.40)	(3.12)	(3.09)	(2.12)
Characteristics	yes	yes	yes	yes	yes	yes	yes
R^2_{adj} [%]	7.29	9.06	5.52	6.41	8.70	10.77	29.29
$\bar{n}$	1868	932	2804	2829	1396	733	49
T	648	324	324	648	648	648	648

Table 8: Continued

Panel D: Alternative Factor Models

	(1) 7F	(2) 3F	(3) 4F	(4) 5F	(5) 5F+UMD	(6) 5F+BAB	(7) M4	(8) Q5
MCRASH	2.69	1.37	1.64	2.22	2.45	2.48	1.72	1.77
	(3.96)	(3.24)	(3.18)	(3.78)	(3.72)	(3.97)	(3.05)	(2.76)
Characteristics	yes	yes	yes	yes	yes	yes	yes	yes
R^2_{adj} [%]	7.29	7.30	7.31	7.29	7.30	7.29	7.19	6.52

Panel A of this table reproduces the univariate portfolio sort results from Table 2 with a value-weighting scheme. We report seven-factor alphas for each of the ten value-weighted portfolios sorted on MCRASH and for the (10)-(1) spread portfolio. We run this exercise for our full sample (all) and for sub-samples that exclude the 1%, 5%, and 10% largest stocks (as measured by their market capitalization at the end of month t). In Panels B to D, we summarize Fama and MacBeth (1973) regressions of future returns on MCRASH. We only report the slope coefficient of MCRASH but we include the same controls as in specification (8) of Table 4, which is the baseline specification (1) in all three panels. In Panel B, we vary the estimation methodology for MCRASH: Columns (2) and (3) are based on the tail probability levels $p = 0.10$ and $p = 0.025$. In columns (4) and (5), we use non-parametric and fully parametric MCRASH-estimates. To obtain the results in columns (6) and (7), we estimate the marginal models with a larger number of observations (up to 500 days and 1000 days). The columns (8) and (9) report results based on alternative GARCH specifications and column (10) relies on estimates from a fully parametric copula specification with DCC dynamics. Details for these estimators are provided in Section IV of the Internet Appendix. Panel C summarizes results for different test samples using the identical regression setup as in Panel B. In columns (2) and (3), we split our sample in two halves (1965 to 1991, 1992 to 2018). In column (4), we include penny stocks in our sample and only require that a stock has 200 valid returns over the last 250 trading days (including zeros). The regressions in columns (5) and (6) are based on stocks with a market capitalization above the 20% and the 50% percentile of the NYSE sample at the end of each month. In column (7), we reproduce our results for the 49 value-weighted industry portfolios as test assets. In Panel D, we calculate MCRASH based on different factor models. Column (1) again displays results for the baseline seven-factor model with MKT, SMB, HML, RMW, CMA, UMD and BAB. In specifications (2) - (8), we calculate MCRASH for the following sets of factors: the standard Fama and French (1993) three-factor model (3F), the Carhart (1997) four-factor model (4F), the Fama and French (2015) five-factor model (5F) without additional factors, six-factor models that add UMD and BAB to the five-factor model (5F + UMD and 5F + BAB), the four-factor model of Stambaugh and Yuan (2017), and the q5 model of Hou et al. (2021). In all regressions, independent variables are winsorized at the 0.005 probability level. We report t-statistics computed using Newey and West (1987) standard errors with 6 monthly lags in parentheses.

results are stable when we estimate MCRASH nonparametrically or use a fully parametric estimation methodology that combines GARCH-skewed-t margins with a Student-t copula (instead of a semiparametric estimation methodology).[33] In specifications (6) and (7), we demonstrate that our results are stable when we extend the rolling estimation horizon of the GARCH processes up to 500 and 1000 days. Finally, in specifications (8), (9), and (10), we document the robustness of our findings when we model the marginal distributions of stocks and risk factors with alternative GARCH processes, i.e., a GJR specification (Glosten et al., 1993) as well as a standard GARCH specification with normally distributed innovations, and when we estimate MCRASH based on a fully parametric DCC-type copula model that explicitly captures time variation in the conditional correlations. The implementation of these alternative estimators is outlined in Section IV of the Internet Appendix.

In accordance with the findings of Figure 6, specifications (2) and (3) of Panel C show that the impact of MCRASH on future returns is stable over the sub-periods from 1965 to 1991 and 1992 to 2018. We also find that the return effect of MCRASH is stable when we relax our liquidity and price filters and run our tests on all stocks that have at least 200 valid returns (including zeros) during the last 250 trading days (see specification 4). In specifications (5) and (6), we observe a positive and statistically significant relationship between MCRASH and future average returns for stocks with a market capitalization above the 20% and the 50% percentile of the NYSE sample at the end of each month. Finally, in specification (7), we employ the 49 value-weighted Fama and French industry portfolios as test assets (instead of using single stocks).[34] For this sample, we also document a positive and statistically significant impact (at the 10% level) of MCRASH on future returns.

[33]The large coefficient estimate for the parametric MCRASH measures in specification (5) is related to a substantially lower cross-sectional dispersion of the parametric measures compared to our more flexible baseline measures and to increased correlations with beta and size included as control variables.

[34]In this case, we control for an industry's market beta, size, book-to-market ratio, momentum, reversal and its max return in month t.

In our baseline setting, we measure MCRASH with respect to a seven-factor model consisting of MKT, SMB, HML, RMW, CMA, UMD and BAB. To analyze whether the MCRASH premium is specific to the selection of these seven risk factors, we examine the relationship between multivariate crash risk and future returns when MCRASH is calculated based on different factor combinations and models. We report the results in Panel D of Table 8.

Specification (1) repeats the baseline specification of model (8) in Table 4, where we regress excess returns in month $t+1$ on MCRASH in month t, controlling for different firm characteristics. In the remaining specifications (2) - (8), we include identical controls but compute MCRASH for different factors sets. Specifically, we compute MCRASH for subsets of our seven factors including the Fama and French (1993) three-factor model, the Carhart (1997) four-factor model, and the standard Fama and French (2015) five-factor model. In addition, we calculate MCRASH for the four-factor model of Stambaugh and Yuan (2017) and the five-factor model of Hou et al. (2021). In all specifications, we observe positive coefficient estimates which are statistically significant at the 1% level. We conclude that the price effect of MCRASH is not specific to the choice of the factors in our baseline analysis. To the contrary, our results suggest that the non-linear risk premium captured by MCRASH is a common feature of many standard factor models.

In summary, this section confirms our main results of a stable positive relationship between MCRASH and the cross-section of average stock returns for all stocks except the largest 1%. Our results do not depend on specific filters for our stock sample, the estimation procedure of MCRASH or a specific multifactor model.

5 Alternative Notions of Multivariate Crash Risk

The previous section documents the existence of a premium for extreme dependence to multiple priced factors as measured by MCRASH. The definition of multivariate crash risk provided in equation (2) that MCRASH relies on is generally broad in nature. In this section, we seek to analyze alternative and more specific notions of multivariate crash risk. We first look at bivariate crash risk in Section 5.1 and examine the price impact of simultaneous factor crashes in Section 5.2.

5.1 Bivariate Crash Risk Measures

We investigate the price impact of a stock's sensitivities to crash events of *individual* risk factors (i.e., a stock's bivariate crash risk). Consistent with the definition of MCRASH in equation (3), we introduce the crash sensitivity of stock i with respect to factor X_j as

$$\text{CRASH}_i^{X_j} = \mathbb{P}[T_p[R_i] \,|\, T_p[X_j]] = \mathbb{P}[R_i \leq Q_p[R_i] \,|\, X_j \leq Q_p[X_j]] \,. \tag{14}$$

We examine the effect of bivariate crash risk on the cross-section of average future stock returns for the seven risk factors in our baseline analysis, i.e., we analyze the return effect of $\text{CRASH}^{\text{MKT}}$, $\text{CRASH}^{\text{SMB}}$, $\text{CRASH}^{\text{HML}}$, $\text{CRASH}^{\text{RMW}}$, $\text{CRASH}^{\text{CMA}}$, $\text{CRASH}^{\text{UMD}}$, and $\text{CRASH}^{\text{BAB}}$. All these measures are computed with the same semiparametric technique applied for the estimation MCRASH. Summary statistics of these measures are provided in Table IA.10 in the Internet Appendix.

Panel A of Table 9 reports the results of Fama and MacBeth (1973) regressions of future excess returns on each of the bivariate crash coefficients. We apply standard firm characteristics as specified in model (8) of Table 4 as control variables, but do not show their coefficients to save space.

Regression (1) documents a strongly positive coefficient estimate of $\text{CRASH}^{\text{MKT}}$ which is statistically significant at the 1% level. This effect is in line with the findings of Chabi-Yo et al. (2018), who identify a stock's systematic exposure to market crashes as an important determinant of future returns. Specifications (2) to (7) display the results for bivariate non-market crash coefficients. We

Table 9: Bivariate CRASH Risk Measures

Panel A: Regressions with Firm Characteristics

	(1) MKT	(2) SMB	(3) HML	(4) RMW	(5) CMA	(6) UMD	(7) BAB
CRASH	0.48	0.41	0.02	0.48	-0.22	0.20	0.31
	(2.72)	(1.68)	(0.07)	(1.71)	(-0.78)	(0.60)	(0.97)
Characteristics	yes	yes	yes	yes	yes	yes	yes
R^2_{adj}[%]	7.31	7.30	7.29	7.29	7.27	7.34	7.28

Panel B: Regressions with MCRASH and Firm Characteristics

	(1) MKT	(2) SMB	(3) HML	(4) RMW	(5) CMA	(6) UMD	(7) BAB
MCRASH	2.35	2.48	2.99	2.53	2.98	2.76	2.62
	(3.31)	(3.38)	(4.32)	(3.59)	(4.18)	(4.24)	(3.72)
CRASH	0.14	0.06	-0.35	0.17	-0.59	-0.17	-0.01
	(0.73)	(0.25)	(-1.04)	(0.58)	(-1.95)	(-0.50)	(-0.04)
Characteristics	yes	yes	yes	yes	yes	yes	yes
R^2_{adj} [%]	7.36	7.36	7.35	7.35	7.34	7.38	7.34

This table presents the results of Fama and MacBeth (1973) regressions of future excess returns over the risk-free rate on bivariate crash risk measures (CRASH). The bivariate CRASH coefficients are estimated for the seven factors that we use for the definition of our baseline MCRASH measure (i.e., MKT, SMB, HML, RMW, CMA, UMD, and BAB). We apply the same estimation procedure for the CRASH coefficients as for MCRASH. See Appendix A.2 for a detailed description. In Panel A, we investigate the impact of each bivariate CRASH measure controlling for firm characteristics as in specification (8) of Table 4 in all regressions. In Panel B, we add MCRASH as explanatory variable to the specifications shown in Panel A. All independent variables are winsorized at the 0.005 probability level. Our return sample covers the period from 1965-01 until 2018-12. We report t-statistics calculated using Newey and West (1987) standard errors with 6 monthly lags in parentheses.

observe that the coefficient estimates on $\text{CRASH}^{\text{SMB}}$ and $\text{CRASH}^{\text{RMW}}$ are also positive and borderline statistically significant at the 10% level. Hence, we find weak evidence of positive premia for a stock's sensitivity to crashes of the size and the momentum factor.

In a next step, we ask whether the premium for MCRASH is absorbed by any of the bivariate crash risk coefficients. To tackle this research question, we include MCRASH in the regressions summarized in Panel A of Table 9. Panel B reports the results. We find that, in each regression, MCRASH shows a positive coefficient estimate and is statistically significant at the 1% level.[35] Interestingly, the inclusion of MCRASH subsumes the impact of $\text{CRASH}^{\text{MKT}}$ as well as the impact of $\text{CRASH}^{\text{SMB}}$ and $\text{CRASH}^{\text{RMW}}$. Focusing on the relationship between bivariate crash risk with the market and multivariate crash risk, we observe that the inclusion of $\text{CRASH}^{\text{MKT}}$ decreases the coefficient estimate of MCRASH only slightly by approximately 13% from 2.69 (see model 8 of Table 4) to 2.35. To the contrary, including MCRASH in the multivariate regression setup drives down the coefficient estimate of $\text{CRASH}^{\text{MKT}}$ by more than 70% of its original magnitude.

Extending our results from Section 4.3, we conclude that a stock's sensitivity to crashes of the market factor cannot explain the premium for MCRASH. Furthermore, we find that the MCRASH premium is not subsumed by any of the non-market crash risk measures. These findings justify the application of MCRASH as a broad and general measure of crash sensitivity that combines "crash exposure to all factors".

5.2 Simultaneous Factor Crashes

A part of a premium for multivariate crash risk could be driven by investor's fear for simultaneous tail events of several risk factors at the same point in time. In this section, we investigate the price impact of such "perfect storm scenarios", which can be formalized as the intersection of the

[35] We also perform bivariate sorts on MCRASH and the CRASH measures for all seven factors. Our results, shown in Table IA.11 of the Internet Appendix, document that the premium for MCRASH (based on excess returns and alphas) is economically and statistically significant when explicitly controlling for each bivariate crash risk measure.

corresponding individual crash events. Again building on our quantile-based definition of univariate crash risk from equation (1), we define a joint crash of the factors $X_{j_1}, \ldots, X_{j_M}$ as

$$T_p^{\text{joint}}[X_{j_1}, \ldots, X_{j_M}] = \bigcap_{k=1}^{M} T_p[X_{j_k}], \tag{15}$$

where $\{j_{k_1}, \ldots, j_{k_M}\}$ is a subset of the indices $\{1, \ldots, N\}$.[36]

In contrast to the general definition of $T_p[\boldsymbol{X}]$ given in (2), which only requires that (at least) *one* factor realizes a left tail return, $T_p^{\text{joint}}[X_{j_1}, \ldots, X_{j_M}]$ requires that *several* factors *jointly* realize a left tail event. Such joint factor crashes are likely to be especially severe for investors, but, by definition, they only occur very rarely.

To measure a stock's exposure to joint factor crashes, we propose a modification of MCRASH that we refer to as JCRASH. JCRASH of asset i to the factors $X_{j_1}, \ldots, X_{j_M}$ at the probability level p is defined as

$$\text{JCRASH}_i^{X_{j_1}, \ldots, X_{j_M}} = \mathbb{P}\Big[T_p[R_i] \,|\, T_p^{\text{joint}}[X_{j_1}, \ldots, X_{j_M}]\Big]. \tag{16}$$

Consequently, this measure corresponds to the conditional probability that stock i realizes a left tail event given that a simultaneous crash of the factors $X_{j_1}, \ldots, X_{j_M}$ occurs.

Due to the rare occurrences of joint crashes for a high number of risk factors, we do not consider the impact of joint crash events for *all* seven factors from our baseline analysis.[37] Instead, we investigate the price impact of JCRASH for selected subsets of the seven factors with $M \leq 5$. Given the importance of the market factor highlighted in the literature and in the previous subsection, we

[36]The probability of such simultaneous tail events is related to the tail copula of $(X_{j_1}, \ldots, X_{j_k})$. For a definition and additional information on the tail copula, see for example Schmid and Schmidt (2007, p. 1129).

[37]To illustrate this problem, suppose that the elements of $(X_1, \ldots, X_M)'$ are independent. Then, it holds that $\mathbb{P}\big[T_p^{\text{joint}}[X_1, \ldots, X_M]\big] = p^M$. In our baseline setting with $p = 0.05$ and seven risk factors, the probability for a joint crash of all factors is $0.05^7 = 0.00000000078125$.

focus on the six factor pairs including MKT. Furthermore, we calculate JCRASH coefficients for the three factors of the Fama and French (1993) model, the four factors of the Carhart (1997) model and the five factors of the Fama and French (2015) model; that is, we investigate the price impact of $\text{JCRASH}^{\text{MKT,SMB}}$, $\text{JCRASH}^{\text{MKT,HML}}$, $\text{JCRASH}^{\text{MKT,RMW}}$, $\text{JCRASH}^{\text{MKT,CMA}}$, $\text{JCRASH}^{\text{MKT,UMD}}$, $\text{JCRASH}^{\text{MKT,BAB}}$, $\text{JCRASH}^{\text{3F}}$, $\text{JCRASH}^{\text{4F}}$, and $\text{JCRASH}^{\text{5F}}$.

Table IA.12 in the Internet Appendix presents summary statistics on the occurrence probabilities of the relevant crash events. These estimates are based on the semiparametric methodology used for the estimation of MCRASH with $p = 5\%$ as the tail probability level. As expected, we find that average probabilities of simultaneous factor crashes, even when focusing on factor pairs including the market, are very low; the only combination for which the average joint crash probability exceeds 1% is the combination of the market and the momentum factor. The average probability estimate for the three-factor (four-factor) model is given by 0.03% (0.01%) and the semiparametric estimates of simultaneous crash probabilities for the five-factor model are always zero during our sample period.

We therefore compute JCRASH based on a fully parametric approach and emphasize that the corresponding estimates have to be seen as model-based extrapolations beyond the range the available data. Specifically, we use a combination of skewed-t GARCH models for the marginal return distributions and a parametric t-copula for the dependence structure. To obtain JCRASH estimates from these models, we rely on the following copula representation

$$\text{JCRASH}_i^{X_{j_1},\ldots,X_{j_M}} = \frac{C_{R_i,X_{j_1},\ldots,X_{j_M}}(p,\ldots,p)}{C_{X_{j_1},\ldots,X_{j_M}}(p,\ldots,p)} \tag{17}$$

with $C_{\boldsymbol{Y}}$ denoting the copula function of the random vector $\boldsymbol{Y}$.[38]

[38]Under Assumption (A1) from Section 2, this result can easily be derived from the definition in equation (16) as shown in Section IV of the Internet Appendix, where we also provide additional details on our implementation.

Results on the relationship between JCRASH and future returns are documented in Table 10. We again include the same firm characteristics as in specification (8) of Table 4. The results in Panel A suggest that joint factor crashes have a positive effect on future average stock returns.[39] The coefficient estimates of $\text{JCRASH}^{\text{MKT,SMB}}$, $\text{JCRASH}^{\text{MKT,HML}}$, $\text{JCRASH}^{\text{MKT,RMW}}$, $\text{JCRASH}^{\text{MKT,CMA}}$, and $\text{JCRASH}^{\text{MKT,UMD}}$ as well as $\text{JCRASH}^{\text{3F}}$ and $\text{JCRASH}^{\text{5F}}$ are all positive and statistically significant at least at the 10% level. Consequently, these results are supporting the notion that investors show strong aversion against "perfect storm scenarios" and assign a premium to stocks which tend to be adversely affected by such simultaneous factor crashes.

Finally, Panel B of Table 10 investigates the stability of the JCRASH premiums when we add MCRASH to our regressions. We find that $\text{JCRASH}^{\text{MKT,SMB}}$, $\text{JCRASH}^{\text{MKT,RMW}}$, $\text{JCRASH}^{\text{MKT,CMA}}$, $\text{JCRASH}^{\text{MKT,UMD}}$, as well as $\text{JCRASH}^{\text{3F}}$ and $\text{JCRASH}^{\text{5F}}$ show a positive and statistically significant impact (at least at the 10% level) on future returns – even when controlling for MCRASH.

We conclude that a stock's sensitivity to simultaneous factor crashes including the market factor provides incremental information on crash risk premia – in addition to the very robust pricing impact of MCRASH documented in this study.

6 Conclusion

This paper examines the relationship between multivariate crash risk and the cross-section of expected stock returns. In a model with multiple priced factors, investors can be averse to crashes of both the market and non-market risk factors and require a premium for stocks that have high multivariate crash risk. We therefore propose a general multivariate crash risk measure, MCRASH, which captures a stock's sensitivity to extreme realizations of all risk factors in an asset pricing

[39] The slightly lower average numbers of observations for the specifications with more than two factors reflect that we do not always obtain valid JCRASH estimates for numerical reasons due to very low event probabilities.

Table 10: Joint Factor Crashes

Panel A: Regressions with Characteristics

	(1) MKT SMB	(2) MKT HML	(3) MKT RMW	(4) MKT CMA	(5) MKT UMD	(6) MKT BAB	(7) 3F	(8) 4F	(9) 5F
JCRASH	1.25	0.59	1.77	0.78	1.33	0.62	0.66	0.49	0.56
	(3.63)	(1.66)	(3.95)	(2.13)	(2.50)	(1.61)	(2.50)	(1.56)	(2.72)
Characteristics	yes	yes	yes	yes	yes	yes	yes	yes	yes
R^2_{adj} [%]	7.54	7.58	7.50	7.60	7.61	7.55	7.57	7.56	7.49
$\bar{n}$	1868	1868	1868	1868	1868	1868	1866	1868	1834

Panel B: Regressions with MCRASH and Characteristics

	(1) MKT SMB	(2) MKT HML	(3) MKT RMW	(4) MKT CMA	(5) MKT UMD	(6) MKT BAB	(7) 3F	(8) 4F	(9) 5F
MCRASH 7F	1.75	2.54	1.91	2.37	1.92	2.50	2.31	2.40	1.88
	(2.82)	(3.99)	(3.02)	(3.65)	(3.06)	(3.61)	(3.60)	(4.11)	(2.84)
JCRASH	1.11	0.44	1.61	0.61	1.10	0.40	0.54	0.35	0.49
	(3.18)	(1.22)	(3.58)	(1.69)	(2.05)	(1.00)	(2.02)	(1.13)	(2.37)
Characteristics	yes	yes	yes	yes	yes	yes	yes	yes	yes
R^2_{adj} [%]	7.59	7.63	7.55	7.65	7.64	7.60	7.62	7.59	7.55
$\bar{n}$	1868	1868	1868	1868	1868	1868	1866	1868	1834

This table reports results on the pricing of simultaneous factor crashes for factor pairs including the market factor as well as for joint crashes of the three Fama and French (1993) factors, the four Carhart (1997) factors and the five Fama and French (2015) factors. We report the results of Fama and MacBeth (1973) regressions of future excess returns over the risk-free rate on exposure to joint factor crashes as measured by JCRASH. To estimate the JCRASH-coefficients, we use fully parametric copula models that combine GARCH skewed-t margins with a t-copula. These models are again estimated with a rolling estimation window of 250 days. In Panel A, we regress future excess returns on JCRASH controlling for the same firm characteristics as in specification (8) of Table 4. In Panel B, we add MCRASH as an explanatory variable. All independent variables are winsorized at the 0.005 probability level. Our return sample covers the period from 1965-01 until 2018-12. $\bar{n}$ corresponds to the average number of stocks per month used in each for the specifications. We report t-statistics calculated using Newey and West (1987) standard errors with 6 monthly lags in parentheses.

model. Using a new expansion of a generic SDF that depends on multiple risk factors, we are able to isolate a tail-related component of an asset's expected return that is increasing in its exposure to multivariate crash risk as measured by MCRASH.

To investigate the validity of this theoretical prediction, we perform an empirical analysis on the cross-section of individual stock returns in the sample period from 1965 to 2018. In line with our theoretical results, we find that MCRASH shows a significantly positive impact on average future stock returns. Specifically, we find that an investment strategy going long the decile portfolio with the highest MCRASH and going short the decile portfolio with the lowest MCRASH in month t yields an average return spread of 4.68% in month $t+1$ with a t-statistic of 3.69. This spread is stable when we control for linear risk factor exposure in time-series regressions. Moreover, Fama and MacBeth (1973) regressions show that the impact of MCRASH on future returns is not explained by factor betas, stock characteristics or market-based downside risk measures.

We contribute to the theoretical and empirical literature on downside and crash risk in asset pricing. Our results suggests that investors care about the multidimensionality of crash risk and that capturing non-linear extreme dependence with well-known factors helps to improve our understanding of the cross-section of expected stock returns. Whether multivariate crash risk also carries a premium in international equity markets and other asset classes, could be an interesting direction for future research on the topic.

A Appendix

A.1 Derivation Extended Linear Model

Plugging the piecewise linear approximation of the projected SDF from (9) and (10) into the pricing equation (7), we obtain

$$\begin{aligned}\mathbb{E}[R_i - R_f] &\approx -(1+R_f)\,\left(\operatorname{cov}[\nabla m(\boldsymbol{x}_c)\cdot \boldsymbol{X}, R_i] + \operatorname{cov}[d_{\text{tail}}\cdot \mathbb{1}(T_p[\boldsymbol{X}]), R_i]\right) &(18)\\ &\approx \sum_{j=1}^{N} \beta_i^{(j)}\,\lambda^{(j)} + \text{Tail}_i^{\boldsymbol{X}} &(19)\end{aligned}$$

with $\beta_i^{(j)}$ and $\lambda^{(j)}$ given in equation (12) and

$$\text{Tail}_i^{\boldsymbol{X}} = -(1+R_f)\operatorname{cov}[d_{\text{tail}}\cdot \mathbb{1}(T_p[\boldsymbol{X}]), R_i]\,. \tag{20}$$

To relate $\text{Tail}_i^{\boldsymbol{X}}$ to our multivariate crash sensitivity measure $\text{MCRASH}_i^{\boldsymbol{X}}$, we approximate the distribution of R_i in its left tail below the p-quantile by a large negative return $r_{\text{tail}} < 0$. We set

$$R_i \approx r_{\text{tail}}\,\mathbb{1}(T_p[R_i]) + R_i\,\mathbb{1}(\overline{T_p[R_i]}) \tag{21}$$

using the notation for the p-tail introduced in equation (1) and denoting the complementary event of A by $\overline{A}$. Based on this approximation, we can expand $\text{Tail}_i^{\boldsymbol{X}}$ as $\text{Tail}_i^{\boldsymbol{X}} \approx \text{Tail}_i^{\boldsymbol{X},0} + \text{Tail}_i^{\boldsymbol{X},1}$, where

$$\begin{aligned}\text{Tail}_i^{\boldsymbol{X},0} &= -(1+R_f)\,d_{\text{tail}}\operatorname{cov}[\mathbb{1}(T_p[\boldsymbol{X}]), \mathbb{1}(T_p[R_i])]\,r_{\text{tail}}, &(22)\\ \text{Tail}_i^{\boldsymbol{X},1} &= -(1+R_f)\operatorname{cov}\Big[d_{\text{tail}}\,\mathbb{1}(T_p[\boldsymbol{X}]), R_i\,\mathbb{1}(\overline{T_p[R_i]})\Big]\,. &(23)\end{aligned}$$

Since we are mainly interested in the price impact of left tail events, we focus on $\text{Tail}_i^{\boldsymbol{X},0}$ and include $\text{Tail}_i^{\boldsymbol{X},1}$ in the pricing error α_i. To rewrite $\text{Tail}_i^{\boldsymbol{X},0}$ in terms of $\text{MCRASH}_i^{\boldsymbol{X}}$, we exploit that the continuity Assumption (A1) implies $\mathbb{P}[T_p[R_i]] = p$ and

$$\begin{aligned} \text{cov}[\mathbb{1}(T_p[\boldsymbol{X}]), \mathbb{1}(T_p[R_i])] =& \mathbb{P}[T_p[R_i] \cap T_p[\boldsymbol{X}]] - \mathbb{P}[T_p[R_i]]\ \mathbb{P}[T_p[\boldsymbol{X}]] \qquad (24)\\ =& \mathbb{P}[T_p[\boldsymbol{X}]]\ \left(\text{MCRASH}_i^{\boldsymbol{X}} - p\right) \qquad (25) \end{aligned}$$

with

$$\text{MCRASH}_i^{\boldsymbol{X}} = \frac{\mathbb{P}[T_p[R_i] \cap T_p[\boldsymbol{X}]]}{\mathbb{P}[T_p[\boldsymbol{X}]]}. \qquad (26)$$

From (22) and (25), we obtain

$$\text{Tail}_i^{\boldsymbol{X},0} = \lambda_{\text{tail}}^{\boldsymbol{X}} \left(\text{MCRASH}_i^{\boldsymbol{X}} - p\right) \qquad \text{with} \qquad \lambda_{\text{tail}}^{\boldsymbol{X}} := -(1 + R_f)\, \mathbb{P}[T_p[\boldsymbol{X}]] \cdot d_{\text{tail}} \cdot r_{\text{tail}}. \qquad (27)$$

Since $r_{\text{tail}} < 0$ and $\mathbb{P}[T_p[\boldsymbol{X}]] > 0$, the sign of $\lambda_{\text{tail}}^{\boldsymbol{X}}$ corresponds to the sign of d_{tail}. Given its definition in equation (10), d_{tail} is non-negative under the convexity of m according to Assumption (A2).

A.2 Estimation of MCRASH

The following estimation procedure is applied for each stock i and each month t in our sample: We collect the returns of asset i denoted by $(r_{i,s}^d)_{s=1,\ldots,250}$ over the most recent 250 trading days and the corresponding factor returns denoted by $(x_{1,s}^d, \ldots, x_{N,s}^d)_{s=1,\ldots,250}$ in a series of $(N+1) \times 1$-vectors $(\boldsymbol{y}_s)_{s=1,\ldots,250}$ with $y_{1,s} = r_{i,s}^d$ and $y_{j+1,s} = x_{j,s}^d$ for $j = 1, \ldots, N$, $s = 1, \ldots, 250$. Based on the sample $(\boldsymbol{y})_{s=1,\ldots,250}$, we estimate MCRASH with the following two-step methodology.

1. To account for volatility clustering, we first fit GARCH(1,1) models for the $N+1$ marginal

distributions (Bollerslev, 1986).[40] We use the specification

$$Y_{i,s+1} = \mu_i + \sigma_{i,s+1}\, Z_{i,s+1} \quad \text{and} \quad \sigma^2_{i,s+1} = \omega_{i,0} + \omega_{i,1}\,(\sigma_{i,s}\, Z_{i,s})^2 + \omega_{i,2}\,\sigma^2_{i,s}, \tag{28}$$

where μ_i, $\omega_{i,0}, \omega_{i,1}, \omega_{i,2} \in \mathbb{R}$, $\omega_{i,0}, \omega_{i,1}, \omega_{i,2} > 0$ and $\omega_{i,1} + \omega_{i,2} < 1$. In line with the continuity assumption in our theory part, we assume independent and identically distributed time series residuals $Z_{i,s+1}$, which follow the skewed-t distribution proposed by Hansen (1994). For each margin, we obtain a series of conditional cdfs $F_{i,s}(y) := \mathbb{P}[Y_{i,s} \leq y | \mathcal{F}_{s-1}]$, where $\mathcal{F}_{s-1}$ denotes the information available at time $s-1$. We use these cdfs to obtain a sample $(\hat{\boldsymbol{u}}_s)_{s=1,\ldots,250}$ of probability integral transforms given by

$$\hat{u}_{i,s} = F_{i,s}(y_{i,s}). \tag{29}$$

2. Relying on the transformed sample $(\hat{\boldsymbol{u}})_{s=1,\ldots,250}$, we estimate MCRASH non-parametrically based on equation (4) using

$$\text{MCRASH}^{\boldsymbol{X}}_{i|t} = \frac{\sum_{s\in\mathcal{V}} \mathbb{1}(\{\hat{u}_{1,s} \leq q_1\}) \cdot \mathbb{1}(\bigcup_{j=2}^{N+1} \{\hat{u}_{j,s} \leq q_j\})}{\sum_{s\in\mathcal{V}} \mathbb{1}(\bigcup_{j=2}^{N+1} \{\hat{u}_{j,s} \leq q_j\})} \tag{30}$$

with $\mathcal{V}$ denoting the set of days with valid returns for all $N+1$ series and q_i denoting the upper p-quantile of the empirical distribution of $(\hat{u}_{i,s})_{s\in\mathcal{V}}$, $i = 1, \ldots, N+1$. The enumerator counts the number of days in the estimation window on which stock i and one of the factors crash together. The denominator corresponds to the total number of days on which at least one of the factors realizes a left tail event.

[40] We exclude missing values for individual series (but include zero returns). We apply the MFE toolbox provided by Kevin Sheppard for the maximum likelihood estimation of the GARCH models.

Table A.1: Variable and Factor Definitions

Variable	Description
MCRASH$^{\boldsymbol{X}}$	a stock's multivariate crash sensitivity to the risk factors $\boldsymbol{X}$ as defined in equation (3); estimated as described in Appendix A.2
CRASH^{X}	a stock's bivariate crash sensitivity to the risk factor X as defined in equation (14); estimated as detailed in Appendix A.2
JCRASH$^{X_{j_1},X_{j_2}}$	a stock's sensitivity to joint crash events of the factors X_{j_1} and X_{j_2}; see equation (16); estimated parametrically as detailed in Appendix A.2 and Section IV of the Internet Appendix
β^{X}	a stock's beta with the risk factor X, see equation (12); estimated with bivariate regressions using a rolling window of 250 days
size	the natural logarithm of a firm's market capitalization in million USD computed as $\lvert\text{SHROUT} \cdot \text{ALTPRC}\rvert/1000$
bm	a firm's book-to-market ratio computed as detailed in Bali et al. (2016, p. 177ff)
mom	a stock's momentum defined as the 11-month cumulative return of the stock over the period $[t-11, t-1]$ multiplied by 100
rev	short term reversal defined as the stock's return in month t multiplied by 100
illiq	a stock's Amihud (2002) illiquidity measure computed as the absolute value of a stock's daily return over the trading volume in dollars averaged over month t
max	a stock's maximum 1-day return in the current month t
β^{down}	downside beta with the market factor as defined in equation (5) of Ang et al. (2006a); estimated non-parametrically with a 250-day rolling window
β^{tail}	tail beta introduced by Kelly and Jiang (2014); we first replicate the construction of the aggregate "tail" risk measure and estimate β^{tail} as a stock's sensitivity on the aggregate tail risk measure in predictive regressions with a rolling window of 120 months; we require at least 36 valid observations
idiovol	idiosyncratic volatility estimated with the standard three-factor model; idiovol is the annualized residual standard error (scaled by 100) from time-series regressions with a rolling window of 250 daily returns
idioskew	idiosyncratic skewness estimated with the three-factor model; idioskew is the sample skewness of the regression residuals from time-series regressions with a rolling window of 250 daily returns
coskew	coskewness with the market factor as defined in equation (6) of Ang et al. (2006a), estimated non-parametrically with a 250-day rolling window

Table A.1: Continued

Variable	Description
cokurt	cokurtosis with the market factor as e.g. defined by Chabi-Yo et al. (2018, p. 1096), estimated non-parametrically with a 250-day rolling window
VaR	5%-Value-at-Risk estimated as the (upper) 5%-quantile of the empirical distribution multiplied by minus one; we use a 250-day rolling estimation window
β^{bear}	bear beta introduced by Lu and Murray (2019); we obtain the AD bear portfolio returns from the authors and estimate β^{bear} using the shrinkage methodology proposed in Lu and Murray (2019)
MKT	Value-weighted CRSP market-return in excess of the risk-free rate, source: KF
SMB	Small-Minus-Big size factor, source: KF
HML	High-Minus-Low value factor, source: KF
RMW	Fama and French (2015) Robust-Minus-Weak profitability factor, source: KF
CMA	Fama and French (2015) Conservative-Minus-Aggressive investment factor, source: KF
UMD	Up-Minus-Down momentum factor, source: KF
BAB	Frazzini and Pedersen (2014) Betting Against Beta factor, source: AQR
LIQ	Pástor and Stambaugh (2003) traded liquidity factor, source: AUTH
QMJ	Asness et al. (2019) Quality-Minus-Junk factor, source: AQR
STR	Short-Term Reversal factor, source: KF
LTR	Long-Term Reversal factor, source: KF
M4SMB	Stambaugh and Yuan (2017) size factor, source: AUTH
M4MGMT	Stambaugh and Yuan (2017) first mispricing factor, source: AUTH
M4PERF	Stambaugh and Yuan (2017) second mispricing factor, source: AUTH
Q5ME	Hou et al. (2015) size factor, source: AUTH
Q5IA	Hou et al. (2015) investment factor, source: AUTH
Q5ROE	Hou et al. (2015) profitability factor, source: AUTH
Q5EG	Hou et al. (2021) expected growth factor, source: AUTH

This table defines the variables and factors used in our empirical analysis. We abbreviate our data sources as follows: KF for Kenneth French's Data Library, AQR for the AQR website and AUTH for the corresponding author's website.

References

Agarwal, Vikas, Stefan Ruenzi, and Florian Weigert, 2017, Tail risk in hedge funds: A unique view from portfolio holdings, *Journal of Financial Economics* 125, 610–636.

Amihud, Yakov, 2002, Illiquidity and stock returns: Cross-section and time-series effects, *Journal of Financial Markets* 5, 31–56.

Ang, Andrew, and Joseph Chen, 2002, Asymmetric correlations of equity portfolios, *Journal of Financial Economics* 63, 443–494.

Ang, Andrew, Joseph Chen, and Yuhang Xing, 2006a, Downside risk, *Review of Financial Studies* 19, 1191–1239.

Ang, Andrew, Robert J. Hodrick, Yuhang Xing, and Xiaoyan Zhang, 2006b, The cross-section of volatility and expected returns, *Journal of Finance* 61, 259–299.

Asness, Clifford S., Andrea Frazzini, and Lasse Heje Pedersen, 2019, Quality minus junk, *Review of Accounting Studies* 24, 34–112.

Asparouhova, Elena, Hendrik Bessembinder, and Ivalina Kalcheva, 2013, Noisy prices and inference regarding returns, *Journal of Finance* 68, 665–714.

Atilgan, Yigit, Turan G Bali, K Ozgur Demirtas, and A Doruk Gunaydin, 2020, Left-tail momentum: Underreaction to bad news, costly arbitrage and equity returns, *Journal of Financial Economics* 135, 725–753.

Bali, Turan G., Nusret Cakici, and Robert F. Whitelaw, 2011, Maxing out: Stocks as lotteries and the cross-section of expected returns, *Journal of Financial Economics* 99, 427–446.

Bali, Turan G., Robert F. Engle, and Scott Murray, 2016, *Empirical Asset Pricing* (Wiley).

Banz, Rolf W., 1981, The relationship between return and market value of common stocks, *Journal of Financial Economics* 9, 3–18.

Barillas, Francisco, and Jay Shanken, 2018, Comparing asset pricing models, *Journal of Finance* 73, 715–754.

Barroso, Pedro, and Pedro Santa-Clara, 2015, Momentum has its moments, *Journal of Financial Economics* 116, 111–120.

Basu, Sanjoy, 1983, The relationship between earnings' yield, market value and return for NYSE common stocks, *Journal of Financial Economics* 12, 129–156.

Beirlant, Jan, Yuri Goegebeur, Jozef Teugels, and Johan Segers, 2004, *Statistics of Extremes*, Wiley Series in Probability and Statistics (John Wiley & Sons, Ltd, Chichester, UK).

Bollerslev, Tim, 1986, Generalized autoregressive conditional heteroskedasticity, *Journal of Econometrics* 31, 307–327.

Bollerslev, Tim, Andrew J. Patton, and Rogier Quaedvlieg, 2021, Realized semibetas: Disentangling "good" and "bad" downside risks, *Journal of Financial Economics* Forthcoming.

Carhart, Mark M., 1997, On persistence in mutual fund performance, *Journal of Finance* 52, 57–82.

Chabi-Yo, Fousseni, 2012, Pricing kernels with stochastic skewness and volatility risk, *Management Science* 58, 624–640.

Chabi-Yo, Fousseni, Stefan Ruenzi, and Florian Weigert, 2018, Crash sensitivity and the cross-section of expected stock returns, *Journal of Financial and Quantitative Analysis* 53, 1059–1100.

Christoffersen, Peter, Vihang Errunza, Kris Jacobs, and Hugues Langlois, 2012, Is the potential for international diversification disappearing? a dynamic copula approach, *Review of Financial Studies* 25, 3711–3751.

Christoffersen, Peter, and Hugues Langlois, 2013, The joint dynamics of equity market factors, *Journal of Financial and Quantitative Analysis* 48, 1371–1404.

Cochrane, John H., 2005, *Asset Pricing* (Princeton University Press, Princeton).

Daniel, Kent, and Tobias J. Moskowitz, 2016, Momentum crashes, *Journal of Financial Economics* 122, 221–247.

Demarta, Stefano, and Alexander J. McNeil, 2007, The t copula and related copulas, *International Statistical Review* 73, 111–129.

Dhaene, J., M. Denuit, Marc Goovaerts, R. Kaas, and D. Vyncke, 2002, The concept of comonotonicity in actuarial science and finance: theory, *Insurance: Mathematics and Economics* 31, 3–33.

Diez De Los Rios, Antonio, and René Garcia, 2011, Assessing and valuing the nonlinear structure of hedge fund returns, *Journal of Applied Econometrics* 26, 193–212.

Dittmar, Robert F., 2002, Nonlinear pricing kernels, kurtosis preference, and evidence from the cross section of equity returns, *Journal of Finance* 57, 369–403.

Drees, Holger, and Xin Huang, 1998, Best attainable rates of convergence for estimators of the stable tail dependence function, *Journal of Multivariate Analysis* 64, 25–46.

Fama, Eugene F., and Kenneth R. French, 1993, Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics* 33, 3–56.

Fama, Eugene F., and Kenneth R. French, 2015, A five-factor asset pricing model, *Journal of Financial Economics* 116, 1–22.

Fama, Eugene F., and James D. MacBeth, 1973, Risk, return, and equilibrium: empirical tests, *Journal of Political Economy* 81, 607–636.

Fan, Yanqin, and Andrew J. Patton, 2014, Copulas in econometrics, *Annual Review of Economics* 6, 179–200.

Frazzini, Andrea, and Lasse Heje Pedersen, 2014, Betting against beta, *Journal of Financial Economics* 111, 1–25.

Galsband, Victoria, 2012, Downside risk of international stock returns, *Journal of Banking and Finance* 36, 2379–2388.

Glosten, Lawrence R., Ravi Jagannathan, and David E. Runkle, 1993, On the relation between the expected value and the volatility of the nominal excess return on stocks, *Journal of Finance* 48, 1779–1801.

Hansen, Bruce E., 1994, Autoregressive conditional density estimation, *International Economic Review* 35, 705.

Hansen, Lars Peter, and Scott F. Richard, 1987, The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models, *Econometrica* 55, 587–613.

Harrison, J. Michael, and David M. Kreps, 1979, Martingales and arbitrage in multiperiod securities markets, *Journal of Economic Theory* 20, 381–408.

Harvey, Campbell R., and Akhtar Siddique, 2000, Conditional skewness in asset pricing tests, *Journal of Finance* 55, 1263–1295.

Hou, Kewei, Haitao Mo, Chen Xue, and Lu Zhang, 2021, An augmented q-factor model with expected growth, *Review of Finance* 25, 1–41.

Hou, Kewei, Chen Xue, and Lu Zhang, 2015, Digesting anomalies: An investment approach, *Review of Financial Studies* 28, 650–705.

Hou, Kewei, Chen Xue, and Lu Zhang, 2020, Replicating anomalies, *Review of Financial Studies* 33, 2019–2133.

Jackwerth, Jens Carsten, 2000, Recovering risk aversion from option prices and realized returns, *Review of Financial Studies* 13, 433–451.

Jang, Jeewon, and Jangkoo Kang, 2019, Probability of price crashes, rational speculative bubbles, and the cross-section of stock returns, *Journal of Financial Economics* 132, 222–247.

Jegadeesh, Narasimhan, 1990, Evidence of Predictable Behavior of Security Returns, *Journal of Finance* 45, 881–898.

Jegadeesh, Narasimhan, and Sheridan Titman, 1993, Returns to buying winners and selling losers: Implications for stock market efficiency, *Journal of Finance* 48, 65–91.

Kelly, Bryan, and Hao Jiang, 2014, Tail risk and asset prices, *Review of Financial Studies* 27, 2841–2871.

Kiriliouk, Anna, Johan Segers, and Laleh Tafakori, 2018, An estimator of the stable tail dependence function based on the empirical beta copula, *Extremes* 21, 581–600.

Lettau, Martin, Matteo Maggiori, and Michael Weber, 2014, Conditional risk premia in currency markets and other asset classes, *Journal of Financial Economics* 114, 197–225.

Levi, Yaron, and Ivo Welch, 2020, Symmetric and asymmetric market betas and downside risk, *Review of Financial Studies* 33, 2772–2795.

Longin, François, and Bruno Solnik, 2001, Extreme correlation of international equity markets, *Journal of Finance* 56, 649–676.

Lu, Zhongjin, and Scott Murray, 2019, Bear beta, *Journal of Financial Economics* 131, 736–760.

Markowitz, Harry, 1959, *Portfolio Selection* (Yale University Press, New Haven).

McNeil, Alexander J., Rüdiger Frey, and Paul Embrechts, 2015, *Quantitative Risk Management: Concepts, Techniques and Tools*, second edition (Princeton University Press, Princeton).

Meine, Christian, Hendrik Supper, and Gregor N. F. Weiß, 2016, Is tail risk priced in credit default swap premia?, *Review of Finance* 20, 287–336.

Newey, Whitney K., and Kenneth D. West, 1987, A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix, *Econometrica* 55, 703–708.

Pástor, Luboš, and Robert F. Stambaugh, 2003, Liquidity risk and expected stock returns, *Journal of Political Economy* 111, 642–685.

Patton, Andrew J., 2004, On the out-of-sample importance of skewness and asymmetric dependence for asset allocation, *Journal of Financial Econometrics* 2, 130–168.

Petkova, Ralitsa, and Lu Zhang, 2005, Is value riskier than growth?, *Journal of Financial Economics* 78, 187–202.

Poon, Ser-Huang, Michael Rockinger, and Jonathan Tawn, 2004, Extreme value dependence in financial markets: Diagnostics, models, and financial implications, *Review of Financial Studies* 17, 581–610.

Pratt, John W., 1964, Risk aversion in the small and in the large, *Econometrica* 32, 122–136.

Roy, A. D., 1952, Safety first and the holding of assets, *Econometrica* 20, 431–449.

Ruenzi, Stefan, and Florian Weigert, 2018, Momentum and crash sensitivity, *Economics Letters* 165, 77–81.

Schmid, Friedrich, and Rafael Schmidt, 2007, Multivariate conditional versions of spearman's rho and related measures of tail dependence, *Journal of Multivariate Analysis* 98, 1123–1140.

Schneider, Paul, Christian Wagner, and Josef Zechner, 2020, Low-risk anomalies?, *Journal of Finance* 75, 2673–2718.

Shumway, Tyler, 1997, The delisting bias in CRSP data, *Journal of Finance* 52, 327–340.

Stambaugh, Robert F., and Yu Yuan, 2017, Mispricing factors, *Review of Financial Studies* 30, 1270–1315.

van Oordt, Maarten R. C., and Chen Zhou, 2016, Systematic tail risk, *Journal of Financial and Quantitative Analysis* 51, 685–705.

Weigert, Florian, 2016, Crash aversion and the cross-section of expected stock returns worldwide, *Review of Asset Pricing Studies* 6, 135–178.

Zhang, Lu, 2005, The value premium, *Journal of Finance* 60, 67–103.

Internet Appendix
Multivariate Crash Risk

Abstract: This Internet Appendix consists of three sections. Section I presents additional empirical results. Section II compares the SDF approximation proposed in Section 2 of the paper to alternative extensions of the standard linear model. In Section III, we describe the distributional assumptions and the preferences for the stylized theoretical example presented in Section 2 of the main text. Section IV presents additional details on the estimation of MCRASH and JCRASH.

I Additional Empirical Results

Table IA.1: The MCRASH-Spread Portfolio and Factor Returns

	avg	std	skew	kurt	min	q5	med	q95	max	sharpe
PF MCRASH	4.72	8.67	0.19	10.21	-14.79	-3.30	0.29	4.08	17.03	0.54
MKT	5.98	15.39	-0.53	4.88	-23.24	-7.21	0.84	7.03	16.10	0.39
SMB	2.65	10.73	0.48	8.20	-16.86	-4.24	0.14	4.91	21.70	0.25
HML	3.80	9.80	0.09	5.03	-11.18	-3.92	0.24	5.28	12.87	0.39
RMW	3.18	7.60	-0.32	15.03	-18.33	-2.71	0.24	3.42	13.33	0.42
CMA	3.40	6.98	0.30	4.60	-6.86	-2.70	0.13	3.47	9.56	0.49
UMD	7.99	14.64	-1.32	13.39	-34.39	-6.50	0.75	6.41	18.36	0.55
BAB	10.09	11.45	-0.49	7.30	-15.68	-4.70	0.99	5.84	15.39	0.88
LIQ	4.70	11.63	0.01	4.00	-12.78	-5.14	0.35	5.93	11.68	0.40
QMJ	4.79	7.86	0.22	5.78	-9.10	-3.01	0.32	4.05	12.41	0.61
STR	5.77	10.77	0.36	8.61	-14.60	-3.46	0.32	5.01	16.21	0.54
LTR	2.82	8.71	0.64	5.53	-7.80	-3.51	0.14	4.72	14.50	0.32
M4SMB	5.66	10.00	0.33	4.96	-11.10	-3.86	0.41	5.02	16.04	0.57
M4PERF	8.18	13.24	-0.09	6.59	-21.45	-5.17	0.67	6.74	18.52	0.62
M4MGMT	7.10	9.90	0.14	4.74	-8.93	-3.82	0.57	5.50	14.58	0.72
Q5ME	3.45	10.61	0.61	8.14	-14.39	-4.57	0.19	5.16	22.14	0.33
Q5IA	4.48	6.51	0.14	4.34	-7.16	-2.48	0.32	3.33	9.24	0.69
Q5ROE	6.57	8.69	-0.71	7.80	-13.83	-3.39	0.65	3.97	10.38	0.76
Q5EG	10.01	6.52	0.21	4.95	-6.29	-1.94	0.74	3.73	10.82	1.53

This table presents risk and performance statistics for our (10)-(1) spread portfolio formed on MCRASH and for the monthly returns of the underlying risk factors, as well as additional factors used for risk adjustments. We report the annualized mean (avg), standard deviation (std), skewness (skew), kurtosis (kurt), minimum return (min) and maximum return (max) as well as the annualized sharpe ratio (sharpe).

Table IA.2: MCRASH-Spread Portfolio – Factor Loadings

	PF MCRASH							
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
alpha [%]	0.44	0.43	0.41	0.47	0.43	0.44	0.39	0.36
	(4.79)	(4.40)	(4.48)	(4.34)	(4.53)	(5.05)	(3.03)	(2.71)
MKT	0.23	0.18	0.24	0.22	0.23	0.24	0.19	0.21
	(7.89)	(4.75)	(7.71)	(6.45)	(8.08)	(8.26)	(4.49)	(5.78)
SMB	-0.03	-0.06	-0.06	-0.04	-0.03	0.03		
	(-0.59)	(-1.16)	(-1.54)	(-0.79)	(-0.61)	(0.62)		
HML	0.12	-0.05	0.09	0.10	0.12	0.18		
	(2.00)	(-0.89)	(1.55)	(1.54)	(2.01)	(3.39)		
RMW	0.06	-0.09	0.03	0.11	0.06	0.03		
	(0.61)	(-0.82)	(0.28)	(1.09)	(0.61)	(0.29)		
CMA	-0.18	-0.27	-0.14	-0.17	-0.18	-0.08		
	(-1.78)	(-2.50)	(-1.37)	(-1.74)	(-1.79)	(-0.76)		
UMD	0.20		0.18	0.20	0.20	0.20		
	(5.53)		(5.10)	(5.89)	(5.25)	(6.47)		
BAB	-0.34		-0.32	-0.34	-0.34	-0.36		
	(-8.32)		(-8.04)	(-8.49)	(-8.36)	(-8.95)		
LIQ			0.01					
			(0.42)					
QMJ				-0.09				
				(-0.82)				
STR					0.01			
					(0.25)			
LTR						-0.21		
						(-3.73)		
M4SMB							-0.04	
							(-0.87)	
M4PERF							0.07	
							(1.09)	
M4MGMT							-0.2	
							(-1.80)	
Q5ME								-0.07
								(-1.31)
Q5IA								-0.21
								(-2.08)
Q5ROE								0.11
								(1.61)
Q5EG								-0.07
								(-0.74)

This table summarizes time-series regressions of the MCRASH spread portfolio returns on the factor returns for selected factor models complementing the results in Table 2.

Table IA.3: Robustness FMB Regressions

Panel A: WLS Regressions with Betas

	future excess return							
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
MCRASH	4.53	5.71	5.09	5.50	5.12	4.88	4.45	4.39
	(3.69)	(5.84)	(5.48)	(5.84)	(5.99)	(5.91)	(5.91)	(5.92)
β^{MKT}		-0.25	-0.22	0.02	0.14	0.17	-0.04	0.03
		(-1.56)	(-1.19)	(0.10)	(0.64)	(0.74)	(-0.15)	(0.14)
β^{SMB}			-0.07	-0.08	0.07	0.08	0.10	0.05
			(-0.64)	(-0.69)	(0.54)	(0.66)	(0.87)	(0.44)
β^{HML}				0.20	0.24	0.11	0.03	0.05
				(1.74)	(1.75)	(0.93)	(0.23)	(0.38)
β^{RMW}					0.19	0.23	0.25	0.27
					(2.49)	(2.94)	(3.75)	(3.73)
β^{CMA}						0.12	0.10	0.07
						(1.29)	(1.14)	(0.92)
β^{UMD}							-0.07	0.03
							(-0.25)	(0.12)
β^{BAB}								0.00
								(0.03)
Intercept	0.17	0.33	0.38	0.30	0.31	0.31	0.33	0.34
	(0.69)	(1.59)	(1.88)	(1.53)	(1.59)	(1.66)	(1.79)	(1.89)
R^2_{adj} [%]	0.34	2.91	4.04	4.70	5.12	5.42	5.88	6.13
$\bar{n}$	2280	2280	2280	2280	2280	2280	2280.00	2280

Table IA.3: Continued

Panel B: WLS Regressions with Characteristics

	future excess return								
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
MCRASH	4.53	5.71	4.81	4.28	3.43	3.28	3.27	2.64	2.74
	(3.69)	(5.84)	(6.41)	(5.54)	(4.63)	(4.45)	(4.38)	(3.68)	(4.53)
β^{MKT}		-0.25	-0.28	-0.19	-0.28	-0.27	-0.31	-0.11	0.16
		(-1.56)	(-1.58)	(-1.12)	(-1.80)	(-1.76)	(-2.02)	(-0.81)	(0.68)
size			0.01	0.01	0.00	0.01	-0.03	-0.09	-0.10
			(0.16)	(0.22)	(0.08)	(0.32)	(-0.60)	(-2.16)	(-2.64)
bm				0.28	0.26	0.28	0.28	0.24	0.21
				(3.57)	(3.44)	(3.62)	(3.46)	(3.06)	(2.98)
mom					0.01	0.01	0.01	0.01	0.01
					(6.35)	(5.78)	(5.34)	(5.26)	(6.58)
rev						-0.04	-0.04	-0.02	-0.03
						(-8.56)	(-8.49)	(-5.10)	(-6.06)
illiq							-0.03	0.01	-0.01
							(-0.52)	(0.27)	(-0.11)
max								-8.65	-7.92
								(-9.78)	(-9.84)
betas	no	no	no	no	no	no	no	no	yes
Intercept	0.17	0.33	0.32	0.09	0.10	0.08	0.41	1.11	1.09
	(0.69)	(1.59)	(0.89)	(0.23)	(0.27)	(0.22)	(1.08)	(3.15)	(3.43)
R^2_{adj} [%]	0.34	2.91	4.35	5.06	6.10	6.78	6.98	7.30	9.09
$\bar{n}$	2280	2280	2280	1911	1910	1910	1868	1868	1868

Table IA.3: Continued

Panel C: OLS Regressions with 7F-Betas

	future excess return							
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
MCRASH	4.33	4.76	4.47	4.59	4.50	4.41	3.75	4.31
	(3.58)	(3.96)	(3.76)	(4.05)	(4.30)	(4.46)	(4.62)	(5.84)
$\beta^{\text{MKT,7F}}$		-0.10	-0.11	-0.13	-0.13	-0.13	-0.08	-0.11
		(-0.91)	(-1.07)	(-1.24)	(-1.18)	(-1.20)	(-0.67)	(-0.90)
$\beta^{\text{SMB,7F}}$			0.04	0.02	-0.01	-0.01	-0.03	-0.01
			(0.58)	(0.32)	(-0.11)	(-0.14)	(-0.39)	(-0.09)
$\beta^{\text{HML,7F}}$				0.07	0.01	0.05	0.12	0.15
				(1.33)	(0.21)	(0.87)	(1.82)	(2.13)
$\beta^{\text{RMW,7F}}$					0.19	0.17	0.16	0.16
					(3.68)	(3.18)	(2.93)	(2.57)
$\beta^{\text{CMA,7F}}$						0.07	0.08	0.10
						(1.30)	(1.38)	(1.54)
$\beta^{\text{UMD,7F}}$							0.06	0.04
							(0.50)	(0.32)
$\beta^{\text{BAB,7F}}$								0.07
								(0.87)
Intercept	0.22	0.27	0.28	0.30	0.35	0.36	0.38	0.37
	(0.88)	(1.36)	(1.41)	(1.55)	(1.85)	(1.92)	(2.10)	(2.06)
	37.83	17.41	15.97	12.21	6.52	5.52	3.65	3.99
R2adj [%]	0.41	1.49	2.25	2.98	3.52	4.14	5.24	6.12
avg n	2280	2280	2280	2280	2280	2280	2280	2280

This table presents additional robustness results on the Fama and MacBeth (1973) shown in Tables 3 and 4. Panel A and B report results for cross-sectional regressions implemented with weighted least squares instead of ordinary least squares. We use the month t gross returns as weights following the methodology proposed by Asparouhova et al. (2013). Panel C shows results for the regression specifications from Table 3 with betas that are estimated from multivariate instead of bivariate regressions. More specifically, we use betas estimated from multivariate regressions with our seven baseline factors. We again use a rolling estimation window with 250 days for the beta estimation.

Table IA.4: Bivariate Portfolio Sorts – MCRASH and Characteristics – Returns

	Control Portfolio					
Control	1	2	3	4	5	avg
β^{MKT}	0.25	0.30	0.40	0.50	0.33	0.36
	(3.67)	(4.24)	(4.88)	(6.11)	(2.77)	(6.15)
size	0.36	0.23	0.24	0.11	0.19	0.23
	(3.71)	(2.09)	(2.36)	(1.24)	(1.85)	(2.93)
bm	0.45	0.34	0.24	0.25	0.10	0.28
	(3.91)	(3.19)	(2.35)	(3.24)	(1.20)	(3.83)
mom	0.34	0.14	0.16	0.02	0.24	0.18
	(3.00)	(1.54)	(1.95)	(0.18)	(2.71)	(2.54)
rev	0.37	0.29	0.25	0.25	0.35	0.30
	(3.27)	(3.28)	(2.82)	(2.20)	(3.56)	(3.95)
illiq	0.21	0.16	0.36	0.29	0.32	0.27
	(1.97)	(1.67)	(3.40)	(2.81)	(3.41)	(3.42)
max	0.23	0.16	0.24	0.26	0.52	0.28
	(3.53)	(2.13)	(2.49)	(2.50)	(3.73)	(3.81)

This table reports excess returns resulting from the bivariate portfolio sorts shown in Table 5.

Table IA.5: Bivariate Portfolio Sorts – MCRASH and Betas

Panel A: Returns

	Control Portfolio					
Control	1	2	3	4	5	avg
β^{MKT}	0.25	0.30	0.40	0.50	0.33	0.36
	(3.67)	(4.24)	(4.88)	(6.11)	(2.77)	(6.15)
β^{SMB}	0.28	0.32	0.29	0.42	0.43	0.35
	(2.98)	(3.71)	(3.66)	(4.55)	(4.24)	(5.39)
β^{HML}	0.35	0.42	0.32	0.31	0.32	0.34
	(3.19)	(4.92)	(4.39)	(4.15)	(3.33)	(5.68)
β^{RMW}	0.44	0.37	0.33	0.30	0.30	0.35
	(4.08)	(4.45)	(4.03)	(3.58)	(3.19)	(5.63)
β^{CMA}	0.39	0.32	0.38	0.31	0.26	0.33
	(3.63)	(3.25)	(4.64)	(4.04)	(3.16)	(5.32)
β^{UMD}	0.26	0.21	0.25	0.38	0.42	0.30
	(2.83)	(3.23)	(3.07)	(4.13)	(3.86)	(5.10)
β^{BAB}	0.36	0.45	0.30	0.28	0.31	0.34
	(3.19)	(5.60)	(3.64)	(4.14)	(3.98)	(6.04)

Panel B: Risk-Adjusted Returns

	Control Portfolio					
Control	1	2	3	4	5	avg
β^{MKT}	0.19	0.28	0.41	0.48	0.31	0.33
	(2.62)	(3.87)	(5.15)	(5.78)	(2.65)	(6.61)
β^{SMB}	0.15	0.26	0.25	0.39	0.48	0.31
	(1.71)	(2.86)	(3.21)	(4.29)	(4.52)	(5.15)
β^{HML}	0.31	0.34	0.24	0.25	0.25	0.28
	(2.79)	(3.60)	(3.44)	(3.36)	(2.66)	(5.03)
β^{RMW}	0.35	0.29	0.26	0.19	0.18	0.25
	(2.85)	(3.37)	(3.56)	(2.44)	(2.15)	(4.71)
β^{CMA}	0.41	0.27	0.29	0.24	0.15	0.27
	(3.68)	(2.90)	(3.69)	(3.22)	(1.91)	(4.98)
β^{UMD}	0.21	0.18	0.21	0.34	0.42	0.27
	(2.07)	(2.68)	(2.58)	(4.34)	(3.73)	(5.05)
β^{BAB}	0.36	0.41	0.22	0.18	0.19	0.27
	(3.28)	(4.53)	(2.74)	(2.62)	(2.29)	(5.30)

This table summarizes the results of bivariate portfolios sorts on MCRASH, in which we control for linear betas to the seven factors MKT, SMB, HML, RMW, CMA, UMD and BAB used for the calculation of MCRASH. Panel A reports (excess) returns of the MCRASH spread portfolios and Panel B shows the corresponding seven-factor alphas.

Table IA.6: Summary Statistics – Downside Risk Measures

	Mean	SD	Skew	Kurt	Min	q5	q25	Med	q75	q95	max
MCRASH	0.08	0.03	0.08	2.81	0.00	0.04	0.06	0.08	0.10	0.13	0.17
β^{down}	1.13	0.70	0.43	5.99	-2.22	0.15	0.66	1.05	1.53	2.37	4.72
β^{tail}	0.18	0.42	0.64	12.95	-2.82	-0.39	-0.02	0.14	0.35	0.89	3.15
idiovol	39.48	20.08	2.37	23.89	9.49	17.70	25.59	35.17	48.66	74.46	261.37
idioskew	0.43	1.20	1.68	22.76	-6.73	-1.12	-0.03	0.36	0.81	2.09	11.08
coskew	-0.12	0.16	0.05	3.33	-0.68	-0.37	-0.22	-0.12	-0.01	0.14	0.51
cokurt	2.11	1.01	-0.09	2.96	-1.69	0.48	1.40	2.12	2.82	3.74	5.10
VaR	0.04	0.02	1.12	5.43	0.01	0.02	0.03	0.04	0.05	0.07	0.13
β^{bear}	0.06	0.41	0.24	4.34	-1.63	-0.57	-0.19	0.05	0.30	0.75	1.97

This table reports summary statistics for the downside risk measures that we control for in our asset pricing tests.

Table IA.7: Regressions with Downside Risk Measures – Alternative Estimation

	future excess return					
	(1)	(2)	(3)	(4)	(5)	(6)
MCRASH	2.04	2.42	2.42	1.84	1.89	1.88
	(2.97)	(3.39)	(3.40)	(2.65)	(2.72)	(2.70)
β^{MKT}	-0.07	-0.09	-0.09	0.15	0.11	0.10
	(-0.52)	(-0.64)	(-0.61)	(1.14)	(0.84)	(0.79)
size	-0.13	-0.11	-0.10	-0.16	-0.17	-0.17
	(-3.40)	(-2.62)	(-2.38)	(-4.09)	(-4.28)	(-4.37)
bm	0.21	0.22	0.21	0.19	0.19	0.20
	(2.70)	(2.74)	(2.61)	(2.48)	(2.42)	(2.51)
mom	0.01	0.01	0.01	0.01	0.01	0.01
	(5.37)	(5.13)	(5.03)	(4.87)	(4.85)	(5.57)
rev	-0.03	-0.02	-0.02	-0.03	-0.03	-0.03
	(-6.96)	(-5.46)	(-5.31)	(-6.49)	(-6.56)	(-6.28)
illiq	0.05	0.04	0.03	0.08	0.06	0.07
	(0.94)	(0.74)	(0.60)	(1.43)	(1.16)	(1.24)
max	1.96	-8.09	-8.22	-3.58	-3.21	-2.54
	(1.74)	(-9.49)	(-9.47)	(-4.65)	(-4.27)	(-3.34)
idiovol_1m	-0.02					
	(-8.74)					
idioskew_5y		-0.16				
		(-4.46)				
coskew_5y			0.00			
			(0.86)			
VaR (par)				-0.25		
				(-9.77)		
ES (par)					-0.18	
					(-10.56)	
vol (par)						-0.36
						(-10.81)
Intercept	1.65	1.31	1.23	1.99	2.04	2.04
	(4.98)	(3.71)	(3.49)	(6.07)	(6.24)	(6.07)
	0.00	0.02	0.05	0.00	0.00	0.00
R^2_{adj} [%]	7.51	7.43	7.48	7.61	7.58	7.60
$\bar{n}$	1868	1770	1770	1868	1868	1868

This table extends the regression results from Table 6 to different estimation methodologies for the included downside risk measures. idiovol_1m refers to idiosyncratic volatility estimates obtained with one month of daily data instead of using 250 days. idioskew_5y corresponds to idiosyncratic skewness estimated with the three-factor model using five years of monthly data. coskew_5y is a coskewness measure obtained by regressing a firm's excess return on the market factor and the market factor squared with five years of monthly data. VaR (par) and ES (par) are parametric 5%-Value-at-Risk and Expected Shortfall estimates derived from the GARCH skewed-t models used for the estimation of MCRASH. vol (par) refers to the corresponding conditional volatility estimates.

Table IA.8: Bivariate Portfolio Sorts – MCRASH and Downside Risk Measures – Returns

	Control Portfolio					
Control	1	2	3	4	5	avg
β^{down}	0.23	0.21	0.41	0.31	0.51	0.33
	(3.69)	(2.83)	(5.08)	(3.27)	(4.24)	(5.36)
β^{tail}	0.36	0.25	0.18	0.17	0.34	0.26
	(2.92)	(2.89)	(2.28)	(1.89)	(3.40)	(3.63)
idiovol	0.14	0.11	0.13	0.16	0.46	0.20
	(2.17)	(1.63)	(1.53)	(1.40)	(3.36)	(2.86)
idioskew	0.30	0.22	0.17	0.26	0.47	0.29
	(3.76)	(2.36)	(1.76)	(2.49)	(4.16)	(3.55)
coskew	0.24	0.31	0.16	0.26	0.21	0.24
	(2.47)	(3.63)	(1.60)	(2.78)	(2.13)	(3.30)
cokurt	0.33	0.29	0.15	0.28	0.20	0.25
	(4.23)	(3.70)	(1.97)	(2.99)	(2.10)	(4.16)
VaR	0.12	0.10	0.13	0.25	0.46	0.21
	(1.96)	(1.53)	(1.70)	(2.40)	(3.25)	(3.35)
β^{bear}	0.48	0.30	0.18	0.32	0.20	0.30
	(3.09)	(1.77)	(1.21)	(1.57)	(1.03)	(2.13)

This table reports excess returns resulting from the bivariate portfolio sorts presented in Table 7.

Table IA.9: Value-Weighted Sorts – Excess Returns

	1	2	3	4	5	6	7	8	9	10	10-1
all	0.39	0.45	0.47	0.48	0.53	0.55	0.52	0.49	0.48	0.55	0.16
	(2.10)	(2.56)	(2.60)	(2.63)	(2.95)	(2.97)	(2.77)	(2.55)	(2.40)	(2.66)	(1.23)
ex 1%	0.39	0.43	0.45	0.48	0.55	0.60	0.59	0.59	0.62	0.66	0.27
	(2.01)	(2.27)	(2.38)	(2.47)	(2.81)	(3.06)	(3.01)	(2.92)	(3.01)	(3.02)	(2.06)
ex 5%	0.41	0.46	0.50	0.54	0.60	0.63	0.62	0.63	0.67	0.70	0.29
	(2.02)	(2.27)	(2.48)	(2.62)	(2.89)	(3.00)	(2.90)	(2.95)	(3.03)	(3.00)	(2.26)
ex 10%	0.45	0.49	0.57	0.60	0.64	0.67	0.66	0.65	0.67	0.73	0.28
	(2.09)	(2.31)	(2.67)	(2.76)	(2.94)	(3.08)	(2.98)	(2.86)	(2.89)	(3.04)	(2.40)
ex 20%	0.43	0.52	0.60	0.64	0.68	0.70	0.69	0.69	0.72	0.75	0.32
	(1.90)	(2.27)	(2.63)	(2.75)	(2.90)	(3.00)	(2.95)	(2.88)	(2.94)	(2.98)	(2.95)

This table reports excess returns for the univariate value-weighted portfolio sorts presented in Panel A of Table 8.

Table IA.10: Alternative Measures of Multivariate Crash Risk – Summary Statistics

	Mean	SD	Skew	Kurt	Min	q5	q25	Med	q75	q95	max
$CRASH^{MKT}$	0.22	0.12	0.40	2.90	0.00	0.03	0.13	0.21	0.30	0.43	0.67
$CRASH^{SMB}$	0.11	0.08	0.78	3.57	0.00	0.00	0.04	0.09	0.15	0.25	0.43
$CRASH^{HML}$	0.05	0.06	1.41	5.25	0.00	0.00	0.01	0.03	0.08	0.16	0.33
$CRASH^{RMW}$	0.05	0.06	1.23	4.67	0.00	0.00	0.01	0.03	0.09	0.16	0.34
$CRASH^{CMA}$	0.04	0.05	1.43	5.17	0.00	0.00	0.00	0.02	0.08	0.14	0.32
$CRASH^{UMD}$	0.11	0.08	0.95	4.35	0.00	0.01	0.05	0.09	0.15	0.25	0.45
$CRASH^{BAB}$	0.04	0.05	1.50	5.60	0.00	0.00	0.00	0.02	0.07	0.14	0.31
$JCRASH^{MKT,SMB}$	0.31	0.11	0.16	2.71	0.04	0.13	0.22	0.31	0.39	0.50	0.66
$JCRASH^{MKT,HML}$	0.24	0.12	0.97	4.71	0.03	0.09	0.15	0.22	0.30	0.46	0.76
$JCRASH^{MKT,RMW}$	0.20	0.10	1.10	4.97	0.02	0.07	0.12	0.18	0.25	0.39	0.67
$JCRASH^{MKT,CMA}$	0.22	0.11	0.92	4.54	0.02	0.08	0.14	0.20	0.28	0.42	0.72
$JCRASH^{MKT,UMD}$	0.24	0.10	0.70	3.69	0.04	0.10	0.17	0.23	0.30	0.43	0.64
$JCRASH^{MKT,BAB}$	0.24	0.10	0.56	3.29	0.03	0.10	0.17	0.23	0.30	0.42	0.62
$JCRASH^{3F}$	0.34	0.15	0.35	3.02	0.03	0.12	0.23	0.33	0.44	0.60	0.83
$JCRASH^{4F}$	0.34	0.14	0.15	2.79	0.03	0.13	0.23	0.33	0.43	0.57	0.74
$JCRASH^{5F}$	0.28	0.18	0.65	3.32	0.01	0.05	0.14	0.26	0.40	0.60	0.85

This table reports summary statistics for the alternative multivariate crash risk measures that we investigate in Section 5.

Table IA.11: Bivariate Portfolio Sorts – MCRASH and CRASH

Panel A: Returns

	Control Portfolio					
Control	1	2	3	4	5	avg
$CRASH^{MKT}$	0.23	0.25	0.16	0.14	0.12	0.18
	(3.56)	(3.97)	(2.78)	(2.28)	(1.67)	(3.66)
$CRASH^{SMB}$	0.36	0.28	0.24	0.25	0.21	0.27
	(4.40)	(3.68)	(3.24)	(3.31)	(2.25)	(3.81)
$CRASH^{HML}$	0.31	0.26	0.24	0.28	0.30	0.28
	(2.94)	(2.57)	(2.41)	(3.19)	(3.80)	(3.12)
$CRASH^{RMW}$	0.29	0.27	0.22	0.26	0.25	0.26
	(2.88)	(2.68)	(2.40)	(3.04)	(3.25)	(3.04)
$CRASH^{CMA}$	0.33	0.33	0.32	0.33	0.26	0.31
	(3.11)	(3.15)	(3.48)	(3.96)	(3.31)	(3.56)
$CRASH^{UMD}$	0.26	0.24	0.18	0.20	0.18	0.21
	(3.55)	(3.53)	(2.88)	(3.23)	(2.35)	(3.68)
$CRASH^{BAB}$	0.25	0.27	0.26	0.31	0.39	0.30
	(2.48)	(2.63)	(2.73)	(3.50)	(4.85)	(3.39)

Panel B: Risk-Adjusted Returns

	Control Portfolio					
Control	1	2	3	4	5	avg
$CRASH^{MKT}$	0.17	0.17	0.13	0.13	0.17	0.15
	(2.91)	(2.33)	(1.74)	(1.88)	(2.27)	(2.78)
$CRASH^{SMB}$	0.26	0.21	0.25	0.29	0.27	0.26
	(3.44)	(3.09)	(3.63)	(4.02)	(3.47)	(4.25)
$CRASH^{HML}$	0.42	0.36	0.34	0.35	0.30	0.35
	(4.78)	(4.33)	(4.07)	(4.64)	(4.07)	(4.82)
$CRASH^{RMW}$	0.38	0.36	0.30	0.31	0.22	0.31
	(4.08)	(3.97)	(3.40)	(3.91)	(2.92)	(4.08)
$CRASH^{CMA}$	0.47	0.47	0.42	0.35	0.24	0.39
	(5.41)	(5.40)	(5.22)	(4.76)	(3.31)	(5.20)
$CRASH^{UMD}$	0.22	0.23	0.19	0.22	0.21	0.22
	(3.06)	(3.23)	(2.90)	(3.42)	(2.82)	(3.87)
$CRASH^{BAB}$	0.34	0.36	0.37	0.37	0.34	0.36
	(4.11)	(4.42)	(4.47)	(4.71)	(4.49)	(4.85)

This table presents the results of bivariate portfolio sorts on MCRASH controlling for bivariate crash risk measures. Panel A presents returns and Panel B shows seven-factor alphas.

Table IA.12: Probabilities of Simultaneous Factor Crashes

	Mean	SD	Min	q5	q25	Med	q75	q95	max
MKT & SMB	0.64	0.74	0	0	0	0.40	0.80	2.40	3.20
MKT & HML	0.28	0.60	0	0	0	0	0.40	1.60	3.60
MKT & RMW	0.38	0.49	0	0	0	2e-3	0.80	1.60	2.00
MKT & CMA	0.16	0.41	0	0	0	0	0	1.20	3.20
MKT & UMD	1.12	0.94	0	0	0.40	1.20	2.00	2.80	3.60
MKT & BAB	0.13	0.27	0	0	0	0	2e-3	0.80	1.60
3F	0.03	0.11	0	0	0	0	0	0.40	0.80
4F	0.01	0.07	0	0	0	0	0	0	0.40
5F	0	0	0	0	0	0	0	0	0

This table presents summary statistics on the probabilities of simultaneous crash events for factor pairs including the market factor as well as the joint crash probabilities of all factors in the Fama and French (1993) three-factor model (3F), the Carhart (1997) four-factor model (4F) and the Fama and French (2015) five-factor model (5F). At the end of each month, the simultaneous crash probabilities are estimated with the semiparametric approach described in Appendix A.2, which is also applied for the computation of MCRASH. We report summary statistics on the resulting estimates over time. Numbers are in percentages.

II Alternative SDF Approximations

Our theoretical analysis on the price impact of MCRASH builds on an extended linear approximation of the SDF shown as equation (9) in Section 2. It is given by

$$m_{L,e}(\boldsymbol{X}) = m_L(\boldsymbol{X}) + \mathbb{1}(T_p[\boldsymbol{X}])\, d_{\text{tail}}(\boldsymbol{X}), \tag{IA.1}$$

where $\mathbb{1}(T_p[\boldsymbol{X}])$ is the indicator function for the multivariate crash event defined in equation (2) and $d_{\text{tail}}(\boldsymbol{X})$ is a linear adjustment term that improves the approximation quality over the tail region.[1]

In our baseline analysis, we choose a constant adjustment d_{tail} that corresponds to $d_{\text{tail}}^{base} \equiv m(\boldsymbol{x}_l) - m_L(\boldsymbol{x}_l)$ with $\boldsymbol{x}_l := \mathbb{E}[\boldsymbol{X} \mid T_p[\boldsymbol{X}]]$. This leads to the following piecewise linear approximation

$$m_{L,e}^{base}(\boldsymbol{X}) = \begin{cases} m(\boldsymbol{x}_c) + \nabla m(\boldsymbol{x}_c) \cdot (\boldsymbol{X} - \boldsymbol{x}_c) & \text{if } \overline{T_p[\boldsymbol{X}]} \\ m(\boldsymbol{x}_l) + \nabla m(\boldsymbol{x}_c) \cdot (\boldsymbol{X} - \boldsymbol{x}_l) & \text{if } T_p[\boldsymbol{X}]. \end{cases} \tag{IA.2}$$

A straightforward extension of this approach would be to use a tail-specific first-order approximation to determine d_{tail}. In particular, we could use the Taylor expansion around $\boldsymbol{x}_l$ given by

$$m_{alt}(\boldsymbol{X}) = m(\boldsymbol{x}_l) + \nabla m(\boldsymbol{x}_l) \cdot (\boldsymbol{X} - \boldsymbol{x}_l) \tag{IA.3}$$

to approximate the SDF over the tail region. This form corresponds to the adjustment term $d_{\text{tail}}^{alt}(\boldsymbol{X}) := m_{alt}(\boldsymbol{X}) - m_L(\boldsymbol{X})$ in equation (IA.1) and the resulting piecewise linear approximation

[1] Here, the "tail region" is understood as the *subset* of the range of $\boldsymbol{X}$ that is implicitly defined by our definition of a multivariate crash event in equation (2). Formally, $T_p := \bigcup_{j=1}^{N} \{x_j \leq Q_p[X_j]\} \subset \mathbb{R}^N$ such that $T_p[\boldsymbol{X}] = \{\boldsymbol{X}(\omega) \in T_p\}$.

is

$$m_{L,e}^{alt}(\boldsymbol{X}) = \begin{cases} m(\boldsymbol{x}_c) + \nabla m(\boldsymbol{x}_c) \cdot (\boldsymbol{X} - \boldsymbol{x}_c) & \text{if } \overline{T_p[\boldsymbol{X}]} \\ m(\boldsymbol{x}_l) + \nabla m(\boldsymbol{x}_l) \cdot (\boldsymbol{X} - \boldsymbol{x}_l) & \text{if } T_p[\boldsymbol{X}]. \end{cases} \tag{IA.4}$$

By comparing (IA.2) and (IA.4), we obtain

$$m_{L,e}^{alt}(\boldsymbol{X}) = m_{L,e}^{base}(\boldsymbol{X}) + \mathbb{1}(T_p[\boldsymbol{X}])\,(\nabla m(\boldsymbol{x}_l) - \nabla m(\boldsymbol{x}_c)) \cdot (\boldsymbol{X} - \boldsymbol{x}_l). \tag{IA.5}$$

This representation shows that $m_{L,e}^{alt}$ can be seen as an extension of our approach, which includes an additional term based on the gradient of the original SDF over the tail region.

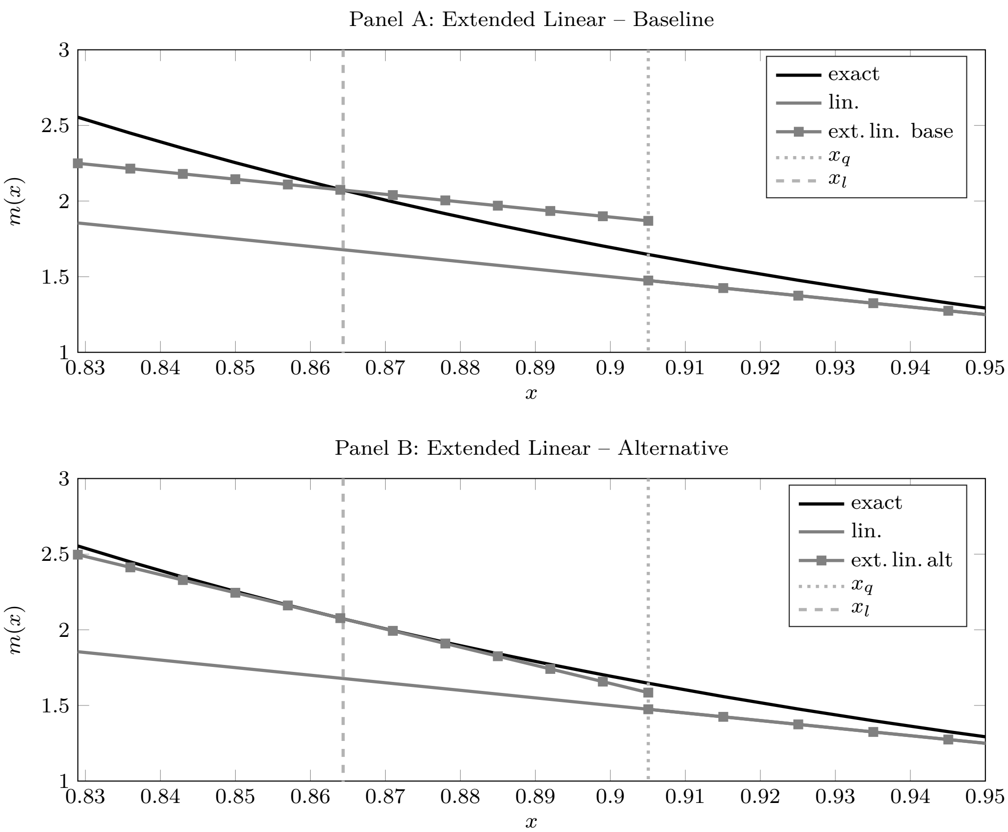
We illustrate both piecewise linear approximations for $N = 1$ in Figure IA.1 building on the same assumptions as the illustration in Figure 2 but focusing on the left tail of X. Panel A shows again the simpler approach with a constant tail adjustment and the resulting reduction in the approximation error compared to the standard linear model. Panel B illustrates the additional improvement that can be attained by modifying the slope of the tail-focussed approximation as formalized by $m_{L,e}^{alt}$.

To understand the pricing implications of $m_{L,e}^{alt}$, we note that using $m(\boldsymbol{X}) \approx m_{L,e}^{alt}(\boldsymbol{X})$ in equation (7) implies

$$\begin{aligned} \mathbb{E}[R_i - R_f] \approx {} & -(1 + R_f)\,\text{cov}\Big[m_{L,e}^{base}(\boldsymbol{X}), R_i\Big] \\ & - (1 + R_f)\,\text{cov}[\mathbb{1}(T_p[\boldsymbol{X}])\,(\nabla m(\boldsymbol{x}_l) - \nabla m(\boldsymbol{x}_c)) \cdot (\boldsymbol{X} - \boldsymbol{x}_l), R_i]\,, \end{aligned} \tag{IA.6}$$

which follows from equation (IA.5). With the alternative tail approximation, we thus obtain the

Figure IA.1: Comparison Piecewise Linear Approximations



This figure illustrates the two piecewise linear approximations of the SDF discussed in Section II of the Internet Appendix. It builds on the same assumptions as the illustration in Figure 2 but focuses on the left tail of the risk factor distribution. Panel A shows our baseline approximation $m_{L,e}^{base}$ defined in equation (IA.2) and Panel B illustrates the alternative approach $m_{L,e}^{alt}$ defined in equation (IA.4).

additional component

$$\Delta\pi_i^{alt} := -(1+R_f)\sum_{j=1}^{N}\left(\frac{\partial m(\boldsymbol{x}_l)}{\partial x_j} - \frac{\partial m(\boldsymbol{x}_c)}{\partial x_j}\right)\operatorname{cov}[\mathbb{1}(T_p[\boldsymbol{X}])\,(X_j - x_{j,l}), R_i] \tag{IA.7}$$

in a stock's risk premium. According to equation (IA.7), the complexity of $\Delta\pi_i^{alt}$ increases in the number of factors N. It shows that N additional tail-specific covariances have to be estimated for the implementation of the alternative model and that these covariances do not only depend on the *occurrence* of the tail event, i.e., the indicator $\mathbb{1}(T_p[\boldsymbol{X}])$, but also on $\mathbb{1}(T_p[\boldsymbol{X}])\,X_j$, $j = 1, \ldots, N$, so that their estimation requires more information on the (joint) distribution of the risk factors over the tail region. Given that it might be difficult to estimate the required covariances and the associated prices of risk precisely, it seems to be an interesting empirical question whether a more flexible functional form in the tail could improve the overall performance of the pricing model.

Besides the piecewise linear approximations discussed before, a further alternative for reducing the errors of a standard linear approximation could be a higher-order Taylor expansion over the entire domain of the projected SDF. While this approach seems promising for $N = 1$, it might be less attractive for models with several factors as the additional complexity from higher-order expansions would grow quickly in the number of factors N. A second-order Taylor expansion would involve the $N \times N$ Hessian matrix and the number of additional covariance terms required to determine the risk premium would grow quadratically with the number of factors.

III Details of the Theoretical Example

For the example presented at the end of Section 2, we use the specific form of the SDF given in equation (6) and assume $N = 2$ with the simple linear mapping function $g(x_1, x_2) = 0.5\,x_1 + 0.5\,x_2$. As a standard choice for the preferences of the representative investor, we rely on power utility

$$u(w) = \frac{w^{1-\eta} - 1}{1 - \eta} \tag{IA.8}$$

with a relative risk aversion (RRA) of $\eta = 5$. Furthermore, we assume $R_f = 0.02$ for the annualized risk-free rate.

We use a flexible copula model for the distribution of (R_i, X_1, X_2), which includes a multivariate normal distribution for the corresponding logarithmic returns as a special case. Instead of directly modeling and simulating discrete returns, we model the joint distribution of *logarithmic* returns to avoid return realizations below minus one. Simulated log-returns are then transformed into discrete returns in line with our theory. For the marginal distributions, we rely on the skewed-t distribution proposed by Hansen (1994), which is also used in our empirical analysis. We assume that the volatility of the (log-)factor returns is equal to 20% per annum and that the annualized volatility of the stock's log-return is equal to 30%. Furthermore, we use identical parameters to calibrate the higher moments of the three marginal distributions. In particular, we choose $\lambda = -0.2$ for the skewness parameters and $\nu = 7$ for the degrees-of-freedom parameters. This implies moderate levels of excess kurtosis (2.41) and skewness (-0.59) for the corresponding log-returns. To determine the location parameters of X_1 and X_2, we numerically solve equation (7) for $R_i = X_1$ and $R_i = X_2$ (simultaneously).[2]

We use the skewed-t-copula introduced by Demarta and McNeil (2007) as dependence model.

[2] Since the covariance underlying our approximations is invariant under deterministic shifts, the location parameter of R_i itself is not required for our simulations.

The skewed t copula is given by

$$C_{st,N}(\boldsymbol{u};\, \boldsymbol{P},\, \nu_c,\, \boldsymbol{\gamma}) = F_{st,N}(q_{st}(u_1;\, \nu_c, \gamma_1), \ldots, q_{st}(u_N;\, \nu_c,\, \gamma_N);\, \boldsymbol{P},\, \nu_c,\, \boldsymbol{\gamma}), \qquad \text{(IA.9)}$$

where $F_{st,N}(\,\cdot\,; \boldsymbol{P},\, \nu_c,\, \boldsymbol{\gamma})$ denotes the cdf of the N-dimensional (generalized hyperbolic) skewed-t distribution with the parameters $\boldsymbol{P}$, ν_c, $\boldsymbol{\gamma}$. $q_{st}(\,\cdot,\, ; \nu_c,\, \gamma_i)$ denotes the quantile function of the corresponding univariate skewed-t distribution with the parameters ν_c and γ_i, $i = 1, \ldots, N$.[3] The skewed t copula nests the standard t copula ($\boldsymbol{\gamma} = \mathbf{0}$) and the Gaussian copula ($\boldsymbol{\gamma} = \mathbf{0}$ and $\nu_c \to \infty$). For our simulations, we set the correlation parameters in $\boldsymbol{P}$ to 0.33 and, we choose $\nu_c = 7$ for the copula's degrees-of-freedom parameter. Furthermore, we assume $\boldsymbol{\gamma} = (-0.2, -0.2, -0.2)'$ for the dependence asymmetry parameter. To obtain the results shown in Panel A and C of Figure 3, we vary the copula asymmetry parameter of the asset in $[-0.85, 0]$. The results in Panel B and D are obtained for values of the copula's degrees-of-freedom parameter between 5 and 15.

In line with our empirical analysis, we simulate monthly returns, i.e., we use $1/12$ and $\sqrt{1/12}$ to rescale the annualized location and scale parameters for the simulation.

[3] Note that the univariate generalized hyperbolic skewed-t distribution is different from the asymmetric skewed-t distribution developed by Hansen (1994).

IV Estimation of MCRASH and JCRASH

We consider the following modifications of the baseline methodology described in Appendix A.2 for the estimation of MCRASH:

- *non-parametric estimation*: We replace the GARCH-based conditional cdfs used in equation (29) with the empirical distribution functions of the sample $(y_{i,s})_{s=1,\ldots,250}$ for $i = 1, \ldots, N+1$.

- *GARCH-normal models*: We assume a standard normal distribution for the GARCH innovations $(Z_{i,s+1})$ in equation (28).

- *GJR-GARCH models*: We replace the GARCH(1,1) dynamics in equation (28) with the GJR specification proposed by Glosten et al. (1993) that allows for a leverage effect.

- *fully parametric estimation*: We apply a parametric approach based on a Student t copula in the second step of our baseline procedure. For each month t and each stock i, we estimate a Student-t-copula model[4] to the corresponding sample of probability integral transforms $(\hat{\boldsymbol{u}}_s)_{s=1,\ldots,250}$. We apply a combination of moment matching and maximum likelihood estimation to determine the copula correlation matrix and the degrees-of-freedom parameter.[5] We then simulate $1,000,000$ realizations from the model and apply the estimator described in equation (30).

- *estimation with time-varying copula*: To account for potential time-variation in the conditional correlation of the asset and factor returns, we estimate t copula models with a time-varying correlation matrix. We implement a simplified version[6] of the DCC-copula models proposed

[4]The t-copula is frequently applied in financial econometrics and risk management, see e.g. Jondeau and Rockinger (2006) or Rosenberg and Schuermann (2006).

[5]This approach is, e.g., proposed by Christoffersen et al. (2012). Similar results are obtained when applying full maximum likelihood estimation.

[6]To keep the rolling window estimations for our large panel tractable, we implement the approach based on the

by Christoffersen et al. (2012) and Christoffersen and Langlois (2013). In particular, we assume the following DCC-style (Engle, 2002; Aielli, 2013) dynamics for the copula correlation matrix:

$$\boldsymbol{P}_{s+1} = \sqrt{\mathrm{diag}(\boldsymbol{Q}_{s+1})}^{-1} \cdot \boldsymbol{Q}_{s+1} \cdot \sqrt{\mathrm{diag}(\boldsymbol{Q}_{s+1})}^{-1}, \tag{IA.10}$$

$$\boldsymbol{Q}_{s+1} = \boldsymbol{S}_c\,(1-\alpha_c-\beta_c) + \alpha_c\,(\bar{\boldsymbol{Z}}_s \cdot \bar{\boldsymbol{Z}}_s') + \beta_c\,\boldsymbol{Q}_s \tag{IA.11}$$

with $\bar{\boldsymbol{Z}}_s = \sqrt{\mathrm{diag}(\boldsymbol{Q}_s)} \cdot \boldsymbol{Z}_t$. $\boldsymbol{S}_c$ is a positive definite $(N+1)\times(N+1)$-matrix, $\alpha_c, \beta_c \in \mathbb{R}$ with $\alpha_c \geq 0$, $\beta_c \geq 0$ and $\alpha_c + \beta_c < 1$. $\boldsymbol{Z}_s$ are the so-called (standardized) "copula shocks" defined as $z_{i,s} = \sqrt{\frac{\nu_c - 2}{\nu_c}}\, q_t(u_{i,s}, \nu_c)$, $i = 1, \ldots, N+1$, with ν_c denoting the degrees-of-freedom parameter of the t copula and q_t as the quantile function of a t distribution (Christoffersen et al., 2012, p. 3718f).

We again use a combination of maximum likelihood estimation and moment matching to determine the parameters of the specification given in the equations (IA.10) and (IA.11) following Christoffersen et al. (2012). To obtain MCRASH forecasts, we simulate from the copula model using the one-step ahead correlation forecast on the last day of our estimation window and apply equation (30).

The copula representation of JCRASH shown in equation (17) can be derived as follows: Under the continuity Assumption (A1) from Section 2, it holds that

$$\mathbb{P}[R_i \leq Q_p[R_i], X_{j_1} \leq Q_p[X_{j_1}], \ldots, X_{j_M} \leq Q_p[X_{j_M}]] = C_{R_i, X_{j_1}, \ldots, X_{j_M}}(p, \ldots, p), \tag{IA.12}$$

symmetric t-copula instead of relying on an asymmetric t-copula model. Furthermore, we do not include a time-trend in the correlation matrix given that we update our model parameters frequently with a rolling estimation window.

and

$$\mathbb{P}[X_{j_1} \leq Q_p[X_{j_1}], \ldots, X_{j_M} \leq Q_p[X_{j_M}]] = C_{X_{j_1},\ldots,X_{j_M}}(p, \ldots, p) \tag{IA.13}$$

with $C_{\boldsymbol{Y}}$ denoting the copula function of the random vector $\boldsymbol{Y}$. From the definition of JCRASH in equation (16), we obtain

$$\begin{aligned}
\mathrm{JCRASH}_i^{X_{j_1},\ldots,X_{j_M}} &= \mathbb{P}[R_i \leq Q_p[R_i] \,|\, X_{j_1} \leq Q_p[X_{j_1}], \ldots, X_{j_M} \leq Q_p[X_{j_M}]] &\text{(IA.14)}\\
&= \frac{\mathbb{P}[R_i \leq Q_p[R_i], X_{j_1} \leq Q_p[X_{j_1}], \ldots, X_{j_M} \leq Q_p[X_{j_M}]]}{\mathbb{P}[X_{j_1} \leq Q_p[X_{j_1}], \ldots, X_{j_M} \leq Q_p[X_{j_M}]]} &\text{(IA.15)}\\
&= \frac{C_{R_i,X_{j_1},\ldots,X_{j_M}}(p, \ldots, p)}{C_{X_{j_1},\ldots,X_{j_M}}(p, \ldots, p)}. &\text{(IA.16)}
\end{aligned}$$

As for the parametric estimation of MCRASH, we apply rolling window estimations to determine the copula parameters. More specifically, we again use a combination of moment matching and maximum likelihood estimation based on the sample of probability integral transforms $(\hat{\boldsymbol{u}}_s)_{s=1,\ldots,250}$ for the relevant returns $(R_i, X_{j_1}, \ldots, X_{j_M})$. To evaluate (IA.16), we then rely on the well-known representation of the t-copula in terms of the multivariate cdf and the univariate quantile functions of the t-distribution, which we implement using the Matlab functions `mvtcdf` and `tinv`.[7] Given the low probability levels that we are interested in, we adjust the absolute error tolerance of the multivariate cdf to 10^{-6} for the analyses with $M \leq 4$ factors and to 10^{-8} for the specification with $M = 5$ factors.

[7]The implementation of the multivariate cdf is based on Genz and Bretz (1999, 2002).

References

Aielli, Gian Piero, 2013, Dynamic conditional correlation: On properties and estimation, *Journal of Business & Economic Statistics* 31, 282–299.

Asparouhova, Elena, Hendrik Bessembinder, and Ivalina Kalcheva, 2013, Noisy prices and inference regarding returns, *Journal of Finance* 68, 665–714.

Carhart, Mark M., 1997, On persistence in mutual fund performance, *Journal of Finance* 52, 57–82.

Christoffersen, Peter, Vihang Errunza, Kris Jacobs, and Hugues Langlois, 2012, Is the potential for international diversification disappearing? a dynamic copula approach, *Review of Financial Studies* 25, 3711–3751.

Christoffersen, Peter, and Hugues Langlois, 2013, The joint dynamics of equity market factors, *Journal of Financial and Quantitative Analysis* 48, 1371–1404.

Demarta, Stefano, and Alexander J. McNeil, 2007, The t copula and related copulas, *International Statistical Review* 73, 111–129.

Engle, Robert, 2002, Dynamic conditional correlation, *Journal of Business & Economic Statistics* 20, 339–350.

Fama, Eugene F., and Kenneth R. French, 1993, Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics* 33, 3–56.

Fama, Eugene F., and Kenneth R. French, 2015, A five-factor asset pricing model, *Journal of Financial Economics* 116, 1–22.

Fama, Eugene F., and James D. MacBeth, 1973, Risk, return, and equilibrium: empirical tests, *Journal of Political Economy* 81, 607–636.

Genz, Alan, and Frank Bretz, 1999, Numerical computation of multivariate t-probabilities with application to power calculation of multiple contrasts, *Journal of Statistical Computation and Simulation* 63, 103–117.

Genz, Alan, and Frank Bretz, 2002, Comparison of methods for the computation of multivariate t probabilities, *Journal of Computational and Graphical Statistics* 11, 950–971.

Glosten, Lawrence R., Ravi Jagannathan, and David E. Runkle, 1993, On the relation between the expected value and the volatility of the nominal excess return on stocks, *Journal of Finance* 48, 1779–1801.

Hansen, Bruce E., 1994, Autoregressive conditional density estimation, *International Economic Review* 35, 705.

Jondeau, Eric, and Michael Rockinger, 2006, The Copula-GARCH model of conditional dependencies: An international stock market application, *Journal of International Money and Finance* 25, 827–853.

Rosenberg, Joshua V., and Til Schuermann, 2006, A general approach to integrated risk management with skewed, fat-tailed risks, *Journal of Financial Economics* 79, 569–614.

cfr working papers are available for download from www.cfr-cologne.de.

2021

No.	Author(s)	Title
21-07	F. Chabi-Yo, M. Huggenberger, F. Weigert	Multivariate Crash Risk
21-06	V. Agarwal, H. Aslan, L. Huang, H. Ren	Political Uncertainty and Household Stock Market Participation
21-05	G. Cici, P. Zhang	On the Valuation Skills of Corporate Bond Mutual Funds
21-04	V. Agarwal, P. Hanouna, R. Moussawi, C.W. Stahel	Do ETFs Increase the Commonality in Liquidity of Underlying Stocks?
21-03	M. Saglam, T. Tuzun, R. Wermers	Do ETFs Increase Liquidity?
21-02	C. Andres, D. Bazhutov, D. Cumming, P. Limbach	The M&A Rumor Productivity Dip
21-01	T. Bali, F. Weigert	Hedge Funds and the Positive Idiosyncratic Volatility Effect

2020

No.	Author(s)	Title
20-14	V. Agarwal, Y. Lu, S. Ray	Are hedge funds' charitable donations strategic?
20-13	M. Jagannathan, W. Jiao, R. Wermers	International Characteristic-Based Asset Pricing
20-12	E. Theissen, L. Zimmermann	Do Contented Customers Make Shareholders Wealthy? - Implications of Intangibles for Security Pricing
20-11	M. Hendriock	Implied Cost of Capital and Mutual Fund Performance
20-10	J. Fink, Stefan Palan, E. Theissen	Earnings Autocorrelation and the Post-Earnings-Announcement Drift – Experimental Evidence
20-09	E. Theissen, C. Yilanci	Momentum? What Momentum?
20-08	V. Agarwal, L. Jiang, Q. Wen	Why Do Mutual Funds Hold Lottery Stocks?
20-07	V. Agarwal, S. Ruenzi, F. Weigert	Unobserved Performance of Hedge Funds
20-06	M. Ammann, S. Fischer, F. Weigert	Factor Exposure Variation and Mutual Fund Performance

20-05	P. Limbach, P. R. Rau, H. Schürmann	The Death of Trust Across the U.S. Finance Industry
20-04	A. Y. Chen, T. Zimmermann	Open Source Cross-Sectional Asset Pricing
20-03	L. Ahnert, P. Vogt, V. Vonhoff, F. Weigert	Regulatory Stress Testing and Bank Performance
20-02	P. Schuster, E. Theissen, M. Uhrig-Homburg	Finanzwirtschaftliche Anwendungen der Blockchain-Technologie
20-01	S. Ruenzi, M. Ungeheuer, F. Weigert	Joint Extreme Events in Equity Returns and Liquidity and their Cross-Sectional Pricing Implications

2019

No.	Author(s)	Title
19-06	S. Greppmair, E. Theissen	Small Is Beautiful? How the Introduction of Mini Futures Contracts Affects the Regular Contract
19-05	G. Cici, M. Hendriock, A. Kempf	Finding your calling: Skill matching in the mutual fund industry
19-04	O. Korn, P. M. Möller, Ch. Schwehm	Drawdown Measures: Are They All the Same?
19-03	G. Cici, M. Hendriock, S. Jaspersen, A. Kempf	#MeToo Meets the Mutual Fund Industry: Productivity Effects of Sexual Harassment
19-02	T. Johann, S. Scharnowski, E. Theissen, C. Westheide, L. Zimmermann	Liquidity in the German Stock Market
19-01	A. Betzer, P. Limbach, P. R. Rau, H. Schürmann	Till death (or divorce) do us part: Early-life family disruption and investment behavior

2018

No.	Author(s)	Title
18-06	M. Goergen, P. Limbach, M. Scholz-Daneshgari	On the Choice of CEO Duality: Evidence from a Mandatory Disclosure Rule
18-05	F. Brochet, P. Limbach, D. Bazhutov, A. Betzer, M. Doumet	Where Does Investor Relations Matter the Most?
18-04	G. Cici, A. Kempf, C. Peitzmeier	Knowledge Spillovers in the Mutual Fund Industry through Labor Mobility
18-03	T. Rischen, E. Theissen	Underpricing in the Euro Area Corporate Bond Market: New Evidence from Post-Crisis Regulation and Quantitative Easing
18-02	S. Lesmeister, P. Limbach, M. Goergen	Trust and Shareholder Voting
18-01	G. Cici, M. Hendriock, A. Kempf	The Impact of Labor Mobility Restrictions on Managerial Actions: Evidence from the Mutual Fund Industry

This document only covers the most recent CFR working papers. A full list can be found at www.cfr-cologne.de.

centre for financial research
cfr/university of cologne
Albertus-Magnus-Platz
D-50923 cologne
fon +49(0)221-470-6995
fax +49(0)221-470-3992
kempf@cfr-cologne.de
www.cfr-cologne.de